SEMI-MARKOV GENERATED POINT PROCESSES AND THE SUPERPOSITION OF FINITE NUMBERS OF INDEPENDENT ERLANG AND HYPER-EXPONENTIAL RENEWAL PROCESSES

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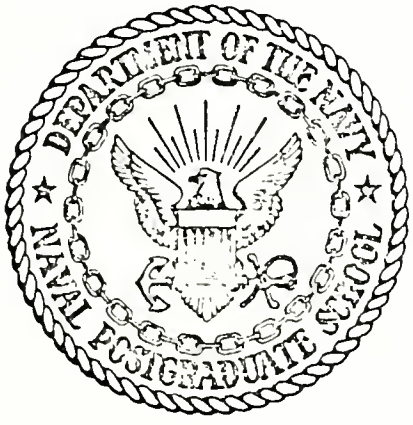

THESIS

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by

Richard Davies Haskell

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(20. ABSTRACT continued)

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Moving average process
Autoregressive process
Mixed moving average/autoregressive process
Hyperexponential renewal process
Semi-Markov Process
Semi-Markov Generated point process
Bispectrum
Polyspectra

Semi-Markov Generated Point Processes and the Superposition of Finite Numbers of Independent Erlang and Hyperexponential Renewal Processes

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I. INTRODUCTION

This thesis is concerned with the analysis and probabilistic properties of point processes which arise from superposing a finite number of renewal processes. Most previous work in this field has considered the case where the number of superposed processes is very large.

The concept of point process is quite general. A point process is a countable set of primitive events defined on a continuum. That is, given a vector space, a point process is any set of vectors defined on the space. The research recorded in this dissertation deals with a specific class of point processes defined on the non-negative real line, (sometimes called a series or stream of events). Moreover, they are univariate point processes or series of events in that events are distinguishable only by where they occur on the real line.

Point processes arise naturally in a wide variety of situations, both deterministic and stochastic in nature. For example, the location of the stars in the universe relative to the earth coordinate and time system is a deterministic point process, while the time, location and magnitude of earthquakes on the surface of the earth may be considered a stochastic point process.

The analysis of a point process is highly dependent upon the nature of the process and the questions to be

answered. For this reason, point processes have been classified into groups within which similar analytic methodology may be applied.

Section A contains a discussion of some of the various classifications of point processes, with examples, followed in Section B by a description of those particular point processes which are the subject of this research.

The scope and limitations of the research are delineated in Sections C and D, with a general overview of the thesis presented in Section E. Some of the more significant results are detailed in Section F.

A summary of previous work relevant to this project is presented in Chapter II.

I.A. CLASSIFICATION OF POINT PROCESSES

Recall that a point process is a set of primitive events defined on a vector space. Restricting attention to those processes which are stochastic in nature, classification is based on the dimension of the vector space and the descriptive properties associated with the primitive events. The following subsections describe some fairly general classes of point processes. The names given for the classes, while not universally accepted, are commonly used (See Lewis, 1972, for further discussion).

I.A.1. The Multidimensional Point Process

The multidimensional point process is a set of events defined on a multidimensional vector space. Typically, although not exclusively, the dimensions are those of space.

An example of such a process might be the list of submarine sinkings in World War II recorded by location. That is, each element of the point process is identified by latitude and longitude only.

I.A.2. The Multivariate Point Process

A multivariate point process is one for which, in addition to its location in a vector space, the events are distinguishable by some other qualitative factor. In the above example of submarine sinkings, to each location vector might be appended the parent alliance (Allied or Axis) of the lost submarine. This process would now be classified as a two dimensional bivariate point process.

I.A.3. The Marked Point Process

A marked point process is one for which the events have quantitative information associated with each event. An obvious example is that of a record of earthquakes which, in addition to the time coordinate, each event has an associated energy level and location vector.

An example of a univariate, unidimensional, marked point process which is easy to visualize is that of arrivals at a supermarket checkout center. The vector space is the positive time line, while the mark is the value of purchases.

I.A.4. Classification by Event Source or Structure

In addition to the above classification list, point processes are categorized by special characteristics of the source of the events, or structure of the process. The definitions are not precise, and some processes may fall in

more than one category. No attempt will be made here to name all the possible categories, but some examples will clarify the idea.

A cluster process is one in which some events are generated by a main or driving process, while other events are in some way triggered by the main events (Lewis, 1964; Vere-Jones, 1970). An example of a cluster process, also known as a branching point process, is the sequence of landing times aboard an aircraft carrier. If several aircraft are assigned a joint mission, the arrival of the first aircraft of the group might be considered the main event, with the arrival times of the remaining members of the group being subsidiary events.

Another classification of point processes according to event source is the superposition of point processes (Cox and Smith, 1954; Cox and Lewis, 1966, Ch. 8; Cinlar, 1972). Under this formulation, the observed events of a point process may have originated in any one of several distinct point processes, but with no indication of which process was the source. This form of point process is the subject of this research, and is explained in more detail below.

A third category is the renewal process to be defined in the next section.

I.B. THE SUPERPOSITION OF RENEWAL POINT PROCESSES

Much of the analysis of point processes is directed toward determining the underlying probabilistic structure

of a point process based on information gained from a statistical sample, or conversely, determining the statistical properties of a sample from assumptions regarding the underlying structure.

A univariate point process which has been the object of extensive study is the renewal process. A renewal process is a univariate point process defined on the non-negative real line for which the separations between adjacent events are independent, identically distributed random variables. It is convenient when discussing unidimensional processes to refer to the events as occurrences in time, and to index the events accordingly. Thus, given events $E_i = t_i$ and $E_j = t_j$, where the indices i and j are taken from the set of integers, and the times are measured from some arbitrary origin, then $t_i < t_j$ whenever $i < j$.

If several independent renewal point processes are operating concurrently, then the union or pooling of the events of all the processes, without regard to which process generated a particular event, forms a superposition of renewal point processes which is itself a univariate point process. Figure I.B.1 shows graphically the superposition of two univariate renewal point processes. The processes which contribute events to the superposition process are called the component processes.

While in the general definition of superposition it is not necessary that the component processes be renewal

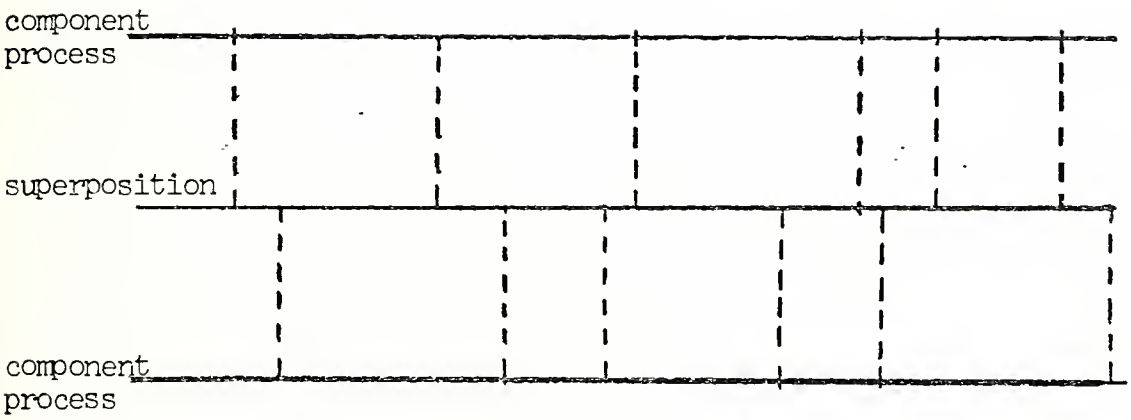


Figure I.B.1. Superposed output of two renewal point processes. This illustrates the basic idea of a superposition process. Intervals in the component processes are independent. In general, the intervals of the superposition process are related.

processes, in this thesis the primary interest is in superposition of independent renewal processes. However, in studying these consideration is given to multivariate or multitype processes called semi-Markov processes; the component processes are dependent renewal processes and their superposition is called a semi-Markov generated point process.

Some examples will clarify the concept and indicate the broad applicability of the superposition model to point processes which occur naturally in science and technology.

Example I.B.1: *Nerve fiber pulses*. Cox and Smith (1954), Cox (1962) and Cox and Lewis (1966) explore the hypothesis that the electrical impulses observed in a neuron are responses to stimulus from several independent sources. Under some distributional assumptions regarding the component processes, including the assumption that they are independent, a method was developed to estimate the number of processes generating impulses in the observed neuron. The methods are very ad hoc and hopefully the results of this thesis will lead to improved methods of analysis.

Example I.B.2: *Failure patterns of series systems*. In reliability theory, a series system is taken to be a piece of equipment which is made up of a combination of component parts such that each component must function in order for the system to function. Thus the system fails each time a component fails. If each component failure results in replacement of that component with a component

having the same failure time distribution as the original, the sequence of system failure times is a univariate superposition process. Lewis (1964) and later Downton (1972) examined this structure with regard to the nature of the stationary failure process as the number of the components grows large.

Example I.B.3: *Resupply of a remote activity.* An activity at great distance from a logistics center is resupplied by a fleet of vehicles which traverse a fixed route. Assuming that the successive round trip transit times for each vehicle may be treated as a renewal process, the arrival pattern of the entire fleet may be represented by a univariate superposition process. Barnett (1970) studied this problem in order to characterize the time between arrivals.

Example I.B.4: *Demand for gas turbine engines for DD 963 class destroyers.* A class of new U. S. Navy destroyers will each be equipped with four gas turbine main propulsion engines. Each time an engine fails or has been operated for a specified time, it must be replaced. The sequence of replacements associated with a particular engine position may be taken to be a renewal process. The demand for engines emanating from all engine positions on all ships is a univariate superposition of renewal processes. This demand pattern is of interest to planners who must arrange for procurement and repair of the gas turbine engines.

Example I.B.5: *Highway traffic analysis.* Automobiles passing a checkpoint on a multilane highway form a superposition process. If a counter is placed so that it records the passage of cars in all lanes, and the stream of cars in each lane is taken to be a point process, the record of the counter is that of a superposition process. In this example, the component processes may be distinctly non-Poisson, non-independent and perhaps non-stationary, indicating the broad applicability of the superposition process.

I.C. RESEARCH OBJECTIVES

The objective of this research has been to characterize in terms of spectral representations and interval distributions the univariate point process resulting from the superposition of a finite number of independent, identically distributed renewal processes with either Erlang or hyperexponential interval distributions. The ultimate goals were to provide the methodology by which a given set of data from a univariate point process might be modeled as the superposition of a finite number of Erlang or hyperexponential renewal processes, the determination of the number of processes superposed being a key question, or to provide means of predicting the behavior of a superposition process based on assumptions regarding the finite number of independent component renewal processes.

A corollary to the direct line of inquiry has involved a broad class of univariate point processes known as

semi-Markov generated point processes. A semi-Markov process may be thought of as a sequence of independent, non-negative random variables sampled in accordance with a Markov chain. If the sequence of times between transitions are taken as interevent times in a univariate point process, a semi-Markov generated point process has been defined. This is the same as the process given in Cox and Lewis (1966, pp. 194) for which the analysis of a two state process is presented. The spectral and distributional characteristics are developed for such processes with finite state space, and the superposition of renewal processes with Erlang or hyperexponential interval distributions is shown to have an equivalent representation as a semi-Markov generated point process. Equivalence here refers to the probabilistic structure of the time between events, and more specifically, the spectral properties.

While a prediction scheme for future behavior of the superposition process based on historical knowledge of the process is not one of the objectives of this work, most practical schemes for prediction have been based on the second order spectral properties which are presented here. See, for instance, Yaglom (1962) or more recently Jowett and Vere-Jones (1972). In this line it should be pointed out that a crucial difference between classical time series (sequences of random variables) and intervals between events in a point process is that the former are almost

always assumed to be normally distributed, while the intervals in a point process are positive random variables and very non-normal.

I.D. RESEARCH LIMITATIONS

In addition to the fact, mentioned above, that the prediction problem has not been examined, other interesting and relevant topics have not been considered. While a major motivation behind the study of superpositions has been estimation of the number of component processes (see Cox and Lewis, 1966, Ch. 8), no effort has been made here to address this problem directly.

The subject of data analysis with regard to spectral and bispectral estimation (see for example Tukey, 1959, and Brillinger and Rosenblatt, 1967) is an important area which complements the current research, but is not included.

In reliability, the failure pattern of a series system may be modeled as a superposition of renewal processes (Example I.B.2). It might be conjectured that a much broader range of system designs may be modeled as semi-Markov generated point processes. In view of the results reported in Chapter V, research in this area might provide an important contribution to reliability theory. It is hoped that the results presented in this thesis will provide the basis for future research along some of the above avenues.

I.E. GENERAL OVERVIEW

This dissertation can be roughly divided into four parts. The first part is direct spectral analysis of the hyperexponential and Erlang superpositions (that is, superpositions of independent, identically distributed renewal point processes with either Erlang or hyperexponential interval distributions). Following is an analysis of the spectral and distributional properties of a semi-Markov generated point process, and the semi-Markov generated point process representations of the Erlang and hyperexponential superpositions.

The third part of the thesis involves study of the marginal interval density of the superposition of renewal processes. Finally, the formulations of the first three parts are developed computationally.

Chapter II contains a summary of definitions, notation and basic results regarding point processes in general which have application to this research project. Also in Chapter II will be found a brief summary of some of the known properties of superposition processes.

Chapter III contains a detailed development of the second order spectral representation of the Erlang superposition process based on the work of Cox and Lewis (1966) and Lewis, et al (1973). Some qualitative results are established regarding the second order interval spectrum, and a relationship with the second order spectrum of a

mixed autoregressive/moving average process is given. An expression is derived for the second order spectrum of counts, relating it to the count spectrum of a component process, and the number of components. Some specific Erlang superpositions are examined in detail.

Chapter IV examines the hyperexponential superposition in much the same way. The development is constrained by the unlimited number of parameters in a general hyperexponential distribution. A two parameter model is studied in detail.

The semi-Markov generated point process is defined and analyzed in Chapter V. This process has been used by Lewis and Shedler (1973) to model page exception processes in multiprogrammed computers. The work of Rudemo (1973) with regard to second order count spectra of semi-Markov generated point processes is summarized. Joint distributions of intervals, serial covariances and higher order joint moments are derived for this process. The second order spectrum of intervals and the interval bispectrum are shown to be functions of the Markov transition matrix and the marginal moments of the process. A method is indicated for obtaining higher order interval spectra of the semi-Markov generated point process.

In Chapter VI, the "method of stages" is used to model hyperexponential and Erlang superposition processes as equivalent semi-Markov generated point processes. It must

again be stressed that equivalence refers to probabilistic structure, and not physical structure. A state of the Markov chain cannot be related to a particular component process, but rather contains information regarding the overall superposition process. The concept of lumpability is employed to reduce the state space associated with the superposition process to a workable size.

In Chapter VII the marginal probability density function of an interval in a process formed by superposing independent renewal processes is shown to be a function of the survivor functions of the component renewal processes, following the development of Cox and Smith (1954). The relationship between the number of processes superposed and the initial value of the marginal probability density function of a superposition interval is shown to have a simple and useful relationship to the number of component processes. The results derived for the superposition of general renewal processes are applied to the Erlang and hyperexponential superpositions.

Numerical testing and tabulation of the results of Chapters III through VII are found in Chapter VIII. A computer oriented discussion covers such diverse topics as the Fast Fourier Transform, and algebraic manipulation compilers such as FORMAC. Alternative methods of solving the various problems are suggested and evaluated quantitatively with regard to response to particular questions and computer efficiency. Several graphs and tables are presented to enable comparison of the various processes, and to indicate

the type of information available from this type of analysis. Included among the graphic displays are second order count and interval spectra, marginal densities, and bispectral representations.

I.E.1. Thesis Organization

Chapters are numbered with Roman numerals, with major sections designated by capital letters. Subsections are indexed within the major sections by Arabic numerals, with further subdivision marked with lower case letters. Thus Subsection III.A.2 is found within Section A of Chapter III.

Theorems, lemmas and examples are numbered sequentially within chapter sections. For example, Theorem III.A.5 is the fifth Theorem of Section III.A. Equations are numbered sequentially within sections and subsections, with chapter reference in Arabic numerals, and enclosed in parentheses. Equation (3.A.2.12) is the twelfth numbered formula in Subsection III.A.2.

The designation of figures follows that of the theorems. That is, Figure V.A.1 is first referenced in Chapter V, Section A.

Appendices follow Chapter VIII and are designated by upper case letters.

I.F. SIGNIFICANT RESULTS

Despite the natural occurrence of superposition processes, there are few analytical results for finite

superpositions (see Cinlar, 1972). In order to fill this gap, this research has concentrated on the study of superpositions of renewal processes having either Erlang or hyperexponential distributions. These particular distributions were chosen because they are reasonably tractable from a computational viewpoint, and they are sufficiently flexible in parameterization that in many cases, empirical data appears to fit one of the two forms.

Several interesting results have been established during the course of this research which may prove valuable to statisticians working with superposition processes.

I.F.1. Spectrum of Intervals and the Relationship of the Superposition Process to a Stationary Mixed Autoregressive/moving Average Process

A mixed autoregressive/moving average process of orders m and n (ARMA (m,n) process) is a sequence of random variables, $\{Z_t\}$, each element of which is equal to a linear combination of the n previous elements of the sequence plus a linear combination of m corresponding random shocks, where the shocks are independent and identically distributed with zero mean and finite variance. A large body of knowledge is available regarding this type of process with respect to identification of the numbers m and n, and estimating the linear coefficients of the model. The major portion of these procedures are based on the serial correlation structure, or equivalently the second order spectral structure of the process. See for example Box and Jenkins (1970).

A superposition process is not a mixed autoregressive/moving average process, nor can it be modeled as one because of the non-negativity of the time between events. However, the second order spectral structure of the intervals of the Erlang and hyperexponential superposition processes is the same as that of a mixed autoregressive/moving average process, opening up much of the time series methodology for application to this class of stochastic processes. This fact is stated as Theorem III.A.2 with proof in Chapter VI.

When a superposition process and an ARMA process have identical second order spectral properties, the processes will be called similar.

I.F.2. The Order of a Similar ARMA Process

With the establishment of the fact that certain types of superposition processes have similar ARMA processes, it was necessary to determine the relationship in detail. Given a sequence of serial correlation coefficients which may be attributed to an ARMA(m,n) process, what can be said about the similar superposition process?

This question was addressed in Chapter III from the inverse point of view. For specific superposition processes, the order of the similar ARMA process was determined. For a class of superposition processes, upper bounds on the ARMA order were specified. In those cases where a final algebraic form for the interval spectrum was determined, the upper

bounds were achieved. However, there is evidence that this will not always be the case.

I.F.3. The Structure of a Semi-Markov Generated Point Process

The semi-Markov process may be thought of as n^2 independent, non-negative random variables sampled in accordance with a Markov chain. If the sequence of times between samplings is taken to be the time between events in a point process, a semi-Markov generated point process has been formed.

The joint structure of intervals in a stationary semi-Markov generated point process is shown in Chapter V to be an easily obtained function of the transition matrix and the n^2 random variables.

Although this formulation has value in its own right, its applicability to this research lies in the fact, demonstrated in Chapter VI, that the classes of superposition processes under study have equivalent representations as semi-Markov generated point processes. That is, the univariate point process generated by either the semi-Markov process, or the superposition process have identical probabilistic structure. Thus, in a reasonably direct way, the spectral structure of all such processes may be determined.

I.F.4. The Inverse of a Characteristic Matrix

In Section 3 it was stated that the spectral representation of the intervals between events in a semi-Markov

generated point process was an easily obtained function of the transition matrix. In fact, the spectrum is a linear combination of terms of the form $(Ix-T)^{-1}$, where this term must be evaluated for all (complex) values of x for which the inverse exists. In Chapter VIII this problem is addressed and three methods of solution are proposed, tested and compared.

II. PRELIMINARY CONCEPTS AND RESULTS

The objective of this chapter is to establish the foundation upon which this research has been based. The work of Cox and Lewis (1966) provided much of the point process technology, some of which has been expanded here to characterize certain superposition processes. Where practicable, the notation of Cox and Lewis has been adopted.

Section A contains definitions and notation, along with the development of some basic results, with regard first to the general stationary, univariate, unideimensional point processes, then for the superposition of renewal point processes. Finally, the special notation peculiar to the particular class of processes under investigation is introduced. Appendix A summarizes the notation found throughout the thesis.

Section B contains a brief summary of known results concerning the superposition of point processes which are of interest in the context of this research.

II.A. PRELIMINARY NOTATION, NOMENCLATURE AND DEFINITIONS

Two distinct avenues of research are represented in this thesis. The primary field is that of the superposition of independent renewal processes. A second class of processes considered are those (univariate) processes generated by the transitions of a semi-Markov process. These univariate point processes are called semi-Markov generated point

processes. The interest in this second class of processes stems from certain distributional properties of the hyper-exponential and Erlang component processes of a superposition which permit the construction of a semi-Markov generated point process equivalent to the superposition of independent renewal processes. The term "equivalent process" means that the probabilistic structure of the point processes generated by either method is the same. It must not be inferred that superpositions of general renewal processes have equivalent semi-Markov generated point processes, nor that every semi-Markov generated point process is equivalent to a superposition process.

While much of the notation and many basic definitions are introduced throughout this thesis, particularly with regard to the semi-Markov process, some will be presented here which are of a general nature.

II.A.1. The Point Process

From here forward, unless otherwise specifically stated, the phrase point process will refer to a stochastic, unidimensional, univariate, regular series of events. That is to say, the events are defined on the non-negative real line, and are distinguishable only by their location on the line. The regularity condition (Daley and Vere-Jone, 1972, Sect. 6) requires that the probability of more than one event in an incremental interval, h, is o(h), where the limit as h tends to zero of o(h)/h is zero. A point process is completely count stationary if the finite dimensional

distributions of counts of events in finite intervals are invariant under translation. (See Cox and Lewis, 1966, Ch. 4.)

A point process has four representations which are physically equivalent in the sense that knowledge of one representation is equivalent to knowledge of the others, although the translation from one representation to another is not necessarily simple. The representations will be referred to as the asynchronous count and interval processes and the synchronous count and interval processes.

Let $\{X_n\}$ be a sequence of iid (independent, identically distributed) non-negative random variables. The sequence $\{X_n\}$ is, by definition, stationary since the finite dimensional distributions

$$\Pr[X_i \leq x_i,\ldots,X_j \leq x_j] = \Pr[X_{i+t} \leq x_i,\ldots,X_{j+t} \leq x_j] \tag{2.A.1.1}$$

for all finite sets of indices $\{i,\ldots,j\}$, and all real valued vectors $(x_i,\ldots,x_j)$. In a renewal theory context $\{X_n\}$ is called an ordinary renewal interval process. In general, if $\{X_n\}$ is a stationary sequence of non-negative random variables giving the times between events in a point process, it will be referred to as the synchronous interval process for the point process. The meaning of the word "synchronous" will become clear in the discussion of arbitrary events, below.

Directly related to the synchronous interval process is the synchronous counting process, $N_f(t)$, which gives the number of events in the interval $(o,t]$, where o is the event time at which the interval x_1 commences. In general $N_f(t)$ is not a process with stationary increments.

There are subtleties connected with the idea of selecting an event to start the counting process which are beyond the scope of this thesis. (See Daley, Vere-Jones, 1972, for more on the Palm-Khinchine theory.)

To define the asynchronous counting process associated with the point process, let the number of events which occur in the interval $(t,t+u)$ be given by $N(t,t+u)$. Assume that t was chosen independently of the stationary interval sequence $\{X_n\}$. Then it can be shown that for any real value h,

$$\Pr[N(t+h,t+h+u) = n] = \Pr[N(t,t+u) = n].$$

This stationary property can be extended and expressed as

$$\Pr[N(t_1,t_1') = n_1,\ldots,N(t_k,t_k' = n_k]$$

$$= \Pr[N(t_1+h,t_1'+h) = n_1,\ldots,N(t_k+h,t_k'+h) = n_k] \qquad (2.A.1.2)$$

for any set of k finite intervals, k = 1,2,..., and for all real h. Thus, N(t,t+u), representing the number of events in an interval of length u from an arbitrary origin, is the asynchronous counting process and will be denoted N(u). Stationarity of a point process usually is taken to mean that N(t,t+u) is a process with stationary increments.

The asynchronous interval process is intimately connected to the asynchronous counting process. Again, taking the basic process to be the synchronous interval sequence $\{X_n\}$, then the arbitrary origin of the asynchronous counting process, N(u), will fall in some interval, say X_j'. The time forward from the arbitrary time of the origin until the next event in the process, denoted W, is governed by the forward recurrence time distribution (Lawrance, 1970)

$$\Pr[W \leq x] = F_W(x) = \mu^{-1} \int_0^x R_X(u)du,$$

where $R_X(u)$ is the marginal survivor function for an element X of the sequence $\{X_n\}$, and $\mu = E[X]$.

Including successive intervals, the resulting sequence $\{W, X_{j+1}', X_{j+2}', \ldots\} = \{W, L_1, L_2, \ldots\}$ is called the asynchronous interval process. In general, the asynchronous interval process is not stationary with respect to the index sequence. The main idea here is that the sampling, performed by choosing an asynchronous origin, picks out an interval X_j' which is length-biased (Cox and Lewis, 1966, Ch. 4) and

doesn't have the same distribution as the intervals $\{X_n\}$. Moreover the effect persists to successive intervals, e.g. X'_{j+1}.

The following expressions will clarify the relationship of the various representations:

$$N(t) < n \Leftrightarrow W+L_1 +\ldots+L_{n-1} > t, \quad n = 1,2,\ldots, \qquad (2.A.1.3)$$

$$N_f(t) < n \Leftrightarrow X_1+X_2 +\ldots+ X_n > t, \quad n = 1,2,\ldots. \qquad (2.A.1.4)$$

An arbitrary event is most easily explained in terms of the asynchronous counting process. Consider the number of events in an interval of length u following an arbitrary event. This may be defined as (Khintchine, 1960)

$$\Pr[N_f(u)=n] = \lim_{h\to 0+} \Pr[N(t+h,t+h+u)=n \mid N(t,t+h)\geq 1] \qquad (2.A.1.5)$$

Equivalently, the interval following an arbitrary event is X_1 of the stationary interval process, with the index origin chosen arbitrarily. The word "synchronous" refers to synchronization with an arbitrary event.

The interval following an arbitrary event will be referred to as an <u>arbitrary interval</u>.

If one ignores the complication caused by the start (asynchronous or synchronous) of the process, it may be observed from (2.A.1.3) and (2.A.1.4) that there are two

stationary stochastic processes, N(t) and $\{X_n\}$, which can represent the process. Consequently, there are two (partly related) second order analyses possible: that of the count process, N(t), and that of the interval process, $\{X_n\}$. Both are useful, the relative utility depending on the process examined. We note that analysis based on N(t) is more fundamental, mainly because data often comes aggregated (as the number of events in a day, for example) without exact times of events. In addition it reflects phenomena in continuous time. However, phenomena which are reflected in serial number will be more marked in the interval properties.

Cox and Lewis (1966, Ch. 4), develop several properties associated with point processes. The following two subsections contain a summary of the pertinent portions of that development.

II.A.2. Properties and Characterizations of the Counting Process

Of interest with respect to the counting representations are the mean-time curves, M(t) and $M_f(t)$, for the synchronous and asynchronous counting processes, respectively, and their intensity functions, m(t) and $m_f(t)$. Also the covariance density function $\gamma(\tau)$, the variance-time curve, V(t), and the second order spectrum of counts for the stationary counting process.

Beginning with the asynchronous counting process,

$$M(t) = E[N(t)] \qquad (2.A.2.1)$$

is the expected number of events in an interval of length t with arbitrary origin. The stationarity of N(t) leads to

$$M(t) = t/\mu \ , \qquad (2.A.2.2)$$

where μ is the expectation of an arbitrary interval. The variance-time curve is defined in a natural way for the asynchronous process:

$$V(t) = E[N^2(t)] - E[N(t)]^2 \ . \qquad (2.A.2.3)$$

The mean-time curve of the synchronous counting process is, using the relation (2.A.1.4),

$$M_f(t) = E[N_f(t)] = \sum_{r=1}^{\infty} Pr[N_f(t) \geq r]$$

$$= \sum_{r=1}^{\infty} F_r(t) \ , \qquad (2.A.2.4)$$

where $F_r(t) = Pr[X_1 + \ldots + X_r \leq t]$, taking $\{X_n\}$ to be the synchronous interval sequence.

The intensity function of a counting process is $m(t) = d/dt\{M(t)\}$ or $m_f(t) = d/dt\{M_f(t)\}$, where $m(t)dt$ may

be thought of as the probability of an event in the infinitesimal interval (t,t+dt). Thus

$$m(t) = \mu^{-1} = m, \tag{2.A.2.5}$$

and, when the derivative exists, from (2.A.2.4)

$$m_f(t) = \sum_{r=1}^{\infty} f_r(t), \tag{2.A.2.6}$$

where $f_r(t)$ is the pdf (probability density function) associated with $F_r(t)$.

To develop the covariance function, let dN(t) denote the number of events in an increment of time. That is,

$$dN(t) = N(t+dt) - N(t).$$

Then, if the process is assumed to be regular,

$$E[dN(t)] = E[(dN(t))^2] = md + o(dt) \tag{2.A.2.7}$$

and, since $(E[dN(t)])^2 = o(dt)$,

$$Var(dN(t)) = mdt + o(dt). \tag{2.A.2.8}$$

Looking now at the joint expectation, for $\tau > 0$

$$E[dN(t)dN(t+\tau)] = Pr[dN(t)=dN(t+\tau)=1] + o(dt^2)$$

$$= mdt\ PR[dN(t+\tau)=1\ dN(t)=1] + o(dt^2)$$

$$= mm_f(t)dt^2 + o(dt^2). \qquad (2.A.2.9)$$

Now define the covariance density as

$$\gamma_+(\tau) = \lim_{dt\to 0} E[(dN(t)-mdt)(dN(t+\tau)-m_f(\tau)dt)]/dt^2$$

$$= \lim_{dt\to 0} [m(m_f(\tau)-m) + o(dt)]$$

$$= m\{m_f(\tau)-m\}\ . \qquad (2.A.2.10)$$

Maintaining consistency with (2.A.2.8) and observing that from the stationarity of $N(t)$, $\gamma_-(-\tau) = \gamma_+(\tau)$, the covariance density function can be defined over the entire real line by

$$\gamma(t) = \begin{cases} m\{m_f(t)-m\}, & t \neq 0, \\ \delta(t)m, & t = 0, \end{cases} \qquad (2.A.2.11)$$

where $\delta(t)$ is the Dirac delta function. This arises essentially from considering the covariance function of the derivative of a jump function, $N(t)$, which exists only in the sense of generalized functions (Daley and Vere-Jones, 1972, Sect. 4).

The Fourier transform of (2.A.2.11) is

$$g(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} \gamma(t) e^{-i\omega t} dt, \quad \omega \geq 0 \qquad (2.A.2.12)$$

where $i = \sqrt{-1}$. Since the process is real and stationary, $m_f(t) = m_f(-t)$, and in the limit as t goes to infinity, $m_f(t)$ tends to m. This permits the alternative form of the count spectrum (Cox and Lewis, 1966, p. 74)

$$g_+(\omega) = m\{1+m_f^*(i\omega)+m_f^*(-i\omega)\}/\pi, \quad \omega \geq 0 \qquad (2.A.2.13)$$

where $m_f^*(s) = \int_0^{\infty} m_f(t) e^{-st} dt$ is the ordinary Laplace transform of $m_f(t)$. The form of the second order count spectrum given by (2.A.2.13) is particularly useful in developing spectral forms in the superposition process.

Further, a very useful relationship exists between the variance-time curve and the second order count spectrum:

$$\lim_{t\to\infty} V'(t) = \pi \lim_{\omega\to 0+} g_+(\omega), \qquad (2.A.2.14)$$

where V'(t) is taken to be the time derivative of the variance-time curve, V(t). That is, the asymptotic slope of the variance-time curve is equal to π times the initial value of the count spectrum.

II.A.3. Properties and Characterizations of the Interval Process

The properties of interest related to the interval process are the expectation, E[X], and variance, Var(X), of

an arbitrary interval; the coefficient of variation,

$$C(X) = \frac{[Var(X)]^{1/2}}{E(X)} , \qquad (2.A.3.1)$$

the auto-correlation sequence $\{\rho_j\}$, and the second order spectral density of the interval process, $f_+(\omega)$.

The autocorrelation sequence, $\{\rho_j\}$ is given by

$$\rho_j = \{E[X_n X_{n+j}] - E[X^2]\}/Var(X), \quad j = 0,\pm1,\pm2,... \qquad (2.A.3.2)$$

The second order interval spectral density is the Fourier transform of the autocorrelation sequence. That is,

$$f_+(\omega) = \{1 + 2 \sum_{j=1}^{\infty} \rho_j \cos j\omega\}/\pi. \quad 0 \leq \omega \leq \pi \qquad (2.A.3.3)$$

An alternative formulation of the spectral density of intervals has proven very useful in this research. Let

$$\phi(z,t) = \sum_{j=0}^{\infty} z^j Pr\{N(t)=j\} \qquad (2.A.3.4)$$

be the probability generating function for N(t), with the Laplace transform $\phi^*(z,s)$. Without going into details (see Cox and Lewis, 1966, p. 77) which require a development not germane to this thesis, the interval spectral density may be determined from

$$f_+(\omega) = \frac{\phi^*(e^{i\omega},0+) + \phi^*(e^{-i\omega},0+)}{\{2\phi^*(0,0+) - E(X)\}}. \qquad (2.A.3.5)$$

This relationship is the basis of one approach to the problem considered in this research.

Finally, a relationship exists between the initial point of the count and interval spectra:

$$g_+(0+) = C^2(X)f_+(0+)/E[X] \ . \qquad (2.A.3.6)$$

II.A.4. The Superposition Process

By superposition process, as used hereafter, is meant the union of the events of several independent renewal point processes. That is, let $N_i(t)$ be a stationary counting process with iid interevent times, for $i = 1,\ldots,p$. Then

$$N^{(p)}(t) = \sum_{i=1}^{p} N_i(t) \qquad (2.A.4.1)$$

is a stationary superposition process. It should be stressed that this is a univariate point process. There is no identification of an event in the superposition with its parent process. An observation of a superposition process is a sequence of times corresponding to the occurrences of events, with no other identifiable characteristics.

In general, a superscript of (p) will indicate that the superscripted symbol refers to a superposition process of p components.

II.A.5. Special Superposition Processes

An Erlang renewal process is a renewal process with interevent times governed by the Erlang pdf,

$$f_X(x) = \lambda^k x^{(k-1)} e^{-\lambda x}/(k-1)!, \quad x \geq 0. \qquad (2.A.5.1)$$

In this thesis, the Erlang pdf will be denoted by $\gamma_x(k,\lambda)$.

When p iid Erlang renewal processes are superposed, the superposition will be called an E(k,p) process. With regard to component processes, the term "iid" means that the processes are independent and have the same probabilistic structure. The scale parameter, λ, is suppressed in this notation. The sequence of intervals resulting from the superposition of p iid Erlang renewal processes will be referred to as the synchronous (asynchronous) E(k,p) interval process, depending on the starting conditions of arbitrary event, or arbitrary time. Denote by $N^{(p)}(t)$ the asynchronous E(k,p) counting process and by $N_f^{(p)}(t)$ the synchronous E(k,p) counting process.

The hyperexponential renewal process has interevent time survivor function

$$R_X(x) = \sum_{j=1}^{k} q_j \exp\{-\lambda_j x\}, \quad x \geq 0, \qquad (2.A.5.2)$$

where $q_j > 0$, $j = 1,\ldots,k$, and $\sum_{j=1}^{k} q_j = 1$. The point process resulting from the superposition of p hyperexponential

renewal processes will be denoted as an $H(k,p;\underline{q},\underline{\lambda})$ process. In this notation, $\underline{q} = (q_1,\ldots,q_k)$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k)$. The usage of $H(k,p;\underline{q},\underline{\lambda})$ will be the same as for $E(k,p)$.

The letters k and p will be reserved for the special usage indicated in this section. Thus, p is the number of component processes in a superposition, and k is the shape parameter of an Erlang distribution, or the number of exponential terms in an hyperexponential distribution. The exact interpretation of k will generally be clear from the context.

II.B. PROPERTIES OF THE SUPERPOSITION PROCESS

Known results regarding superposition processes may be divided roughly into three categories. The first category is that of asymptotic and Poisson related properties, and is covered in Subsection II.B.1. The second category, dealing with the counting process and the marginal interval is covered in Subsection II.B.2. Finally, the joint interval and spectral properties are listed in Subsection II.B.3.

II.B.1. The Asymptotic and Poisson Related Properties of a Superposition Process

The relationship of the Poisson process and superposition process is well covered by Cinlar (1972). The gist of the results presented are

a. Under fairly broad conditions, if a stationary renewal process is the superposition of stationary renewals, each is Poisson.

b. As the number of component processes is increased, with the time scale adjusted so that the mean time between events in the superposition is held constant, the resultant process is approximately Poisson. There are some subtleties regarding this asymptotic result which will not be considered here as they have no bearing on this research. Essentially, the limit holds if no component process contributes inordinately to the total number of events.

II.B.2. Count and Arbitrary Interval Properties of the Superposition Process

Several properties of the superposition of independent, stationary renewal processes are additive, and require little elaboration. Let $N_i(t)$, $i = 1,\ldots,p$, be a set of independent, asynchronous renewal counting processes, with $N^{(p)}(t) = \sum_{i=1}^{p} N_i(t)$. Clearly, the mean-time and variance-time curves are

$$M^{(p)}(t) = \sum_{i=1}^{p} M_i(t) \qquad (2.B.2.1)$$

and

$$V^{(p)}(t) = \sum_{i=1}^{p} V_i(t). \qquad (2.B.2.2)$$

For ease of notation, suppose the component processes are probabilistically identical, where M(t) and V(t) denote the common forms for the component processes. The, if $X^{(c)}$ is an arbitrary interval in a component process, and $X^{(p)}$ is

an arbitrary superposition interval,

$$E[X^{(p)}] = E[X^{(c)}]/p. \qquad (2.B.2.3)$$

The pdf of an arbitrary superposition interval, first given by Cox and Smith (1954), is (assuming iid component processes)

$$f^{(p)}(t) = -\frac{d}{dt}\{R_c(t)(\mu^{-1}\int_t^{\infty} R_c(u)du)^{p-1}\}, \qquad (2.B.2.4)$$

where $R_c(t)$ is the survivor function of the component process interval and $\mu = E[X^{(c)}]$. Development of this form for particular processes is presented in Chapter VII of this dissertation.

Cox and Lewis (1966) give the result

$$m_f^{(p)}(t) = m_f(t) + (p-1)m \qquad (2.B.2.5)$$

based on the idea that an arbitrary event in the superposition process occurs at an arbitrary time in all but one of the component processes.

Enns (1970) determined that for the superposition of iid renewal processes, if the log survivor function is concave (convex), then so is the log survivor function of the superposition process. In the terminology of reliability theory, if the component processes have monotone hazard rate,

then the hazard rate of the superposition is monotone in the same direction.

II.B.3. Second Order Count Properties of the Superposition Process

Taking the covariance function of a stationary counting process as in (2.A.2.10), and using (2.B.2.1) and (2.B.2.5) yields the covariance function of the stationary counting process of the superposition

$$\begin{aligned}\gamma_+^{(p)}(t) &= m^{(p)}(t)\{m_f^{(p)}(t) - m^{(p)}(t)\} \\ &= pm\{m_f(t) + (p-1)m - pm\} \\ &= p\gamma_+(t). \end{aligned} \tag{2.B.3.1}$$

Similarly, taking the Laplace transform of (2.B.2.5) and substituting in the expression for the count spectrum of a stationary process (2.A.1.10) gives

$$\begin{aligned}g_+^{(p)}(\omega) &= pm\{1+m_f^*(i\omega)+m_f^*(-i\omega)\}/\pi \\ &= pg_+(\omega). \end{aligned} \tag{2.B.3.2}$$

These results have the following interesting interpretation. If the component processes are iid renewal processes, and if $g_+^{(p)}(\omega)$ and p are known, then so is $g_+(\omega)$.

This in turn determines the interval properties of the component processes. Thus, in theory everything is known about the process.

If p is not known, however, it is precisely this linearity, or additivity, that makes $g_+^{(p)}(\omega)$ relatively uninformative.

Note that the intervals in the superposition process are correlated and their structure is complicated and difficult to find in practice. However, it is this complex structure that makes their investigation (particularly second order properties) worthwhile for clues to the underlying nature of the superposition process.

II.B.4 Second Order Interval Properties of the Superposition Process

There is, in principal, a means of determining the second order interval spectrum of the superposition process. From the definition of the superposition of independent, identically distributed renewal processes, and (2.A.3.4), it is clear that the generating function of the superposed counting process is

$$\phi^{(p)}(z,t) = \phi^{p}(z,t). \qquad (2.B.4.1)$$

Now, $\phi(z,t)$ is the inverse Laplace transform of $\phi^{*}(z,s)$ which, for a renewal process, is

$$\phi^*(z,s) = \frac{1 - f^*(s)}{s\{1 - zf^*(s)\}} , \qquad (2.B.4.2)$$

given by Lewis, et al (1973), where $f^*(s)$ is the Laplace transform of the renewal interval pdf. The Laplace transform of (2.B.4.1) yields the information required in (2.A.3.5) to determine $f_+^{(p)}(\omega)$.

The procedure outlined above is explained in more detail in Chapter III. The inverse Laplace transform of (2.B.4.2) is not a trivial problem, nor is the expansion of $\phi^p(z,t)$ once $\phi(z,t)$ has been determined. In the case of the E(k,p) process, Lewis, et al (1973) developed a computational form which is the basis for a major portion of this research.

The joint density function of several intervals in the superposition process was derived by Lawrance (1973). The basic argument can be illustrated in terms of the joint density of two adjacent intervals in the synchronous interval sequence. Conditioning on the first interval starting with an event in process 1 at t = 0, the probability that the first interval is terminated in (t_1,t_1+dt) by an event in process 1 and the second interval is terminated in (t_2,t_2+dt) by an event in process 2 is

$$f_1(t_1)R_2(t_2)R_1(t_2-t_1)(\mu_2)^{-1} , \qquad (2.B.4.3)$$

where $f_i(t)$, $R_i(t)$ and μ_i are respectively the pdf, survivor function and expectation of an arbitrary interval in component process i, i = 1,2. Taking all combinations of processes responsible for an event, and starting conditions, expressions of the form (2.B.3.5) are added to give the joint density of the adjacent intervals in the superposition process. In the case of two iid renewal processes, the joint density of two adjacent intervals in the superposition is

$$f^{(2)}_{X_1,X_2}(x_1,x_2) = f(x_1)f(x_2) \int_{x_1+x_2}^{\infty} R(u)du/\mu$$

$$+ R(x_1)f(x_1+x_2)R(x_2)/\mu$$

$$+ f(x_1)R(x_1+x_2)R(x_2)/\mu$$

$$+ R(x_1)f(x_2)R(x_1+x_2)/\mu \ .$$

Lawrance (1973) used this procedure to determine serial correlations of one, two and three lags for certain processes. In Chapter V of this thesis, the joint density function for intervals of a semi-Markov generated point process is derived, and used in Chapter VI to derive joint information regarding the intervals of the E(k,p) and $H(k,p;\underline{q},\underline{\lambda})$ processes.

III. SECOND ORDER SPECTRAL ANALYSIS OF THE E(k,p) INTERVAL PROCESS

The second order spectra of counts and intervals of a point process are available from easily derived formulae (Cox and Lewis, 1966, Ch. 4). Lewis, et al (1973) have established a computational method for determining the interval spectrum of an E(k,p) process. While the computational method is unwieldy for even small values of k and p, its application provides considerable insight into the use of the interval spectrum in the analysis or modelling of a point process.

This chapter deals at length with the second order interval spectrum of the E(k,p) process, demonstrating some general results. The interval spectrum provides a natural analytic tool for the superposition because of the cyclic nature of the process. For example, if three processes are superposed, approximately every third event comes from process 2. In Section A, the general form of the spectrum is established. It is shown constructively that the spectrum is the ratio of finite degree polynomials in $\cos(\omega/k)$, where ω is the argument of the spectrum. A theorem is proposed which relates the E(k,p) interval spectrum to that of a stationary, finite, mixed autoregressive/moving average process.

The special cases of the E(2,p) and E(4,p) processes are considered in Section B. Both processes are shown to

have spectra expressible as the ratio of finite polynomials in cos ω, and an upper bound on the order of the polynomials is established.

Section C gives the method for determining upper bounds for the orders of the ARMA process which is similar to an E(k,p) process.

The second order spectrum of counts is considered in Section D. The results of Cox and Lewis (1966, Ch. 4 and 8) with regard to the count spectrum is applied to the Erlang superposition process. The initial point of the count spectrum is determined for the E(k,p) process. Section E contains some remarks regarding the use of the material of Chapter III along with the methodology proposed by Cox and Lewis (1966) for the characterization of point processes and Box and Jenkins (1970) for ARMA model identification.

III.A. THE INTERVAL SPECTRUM OF AN E(k,p) PROCESS

We recall again the idea that in a rough sense, the spectrum of intervals will be more informative in analyzing superpositions than the spectrum of counts. This is because the interval spectrum relates to the phenomena related to serial number and on the average, every p-th event comes from a given process. This serial number effect is a "hidden periodicity" in the data. In this section two methods of direct computation of the second order spectrum of intervals of an E(k,p) process are discussed. First, the method of Lewis, et al (1973) which has been used

successfully to generate numerical representations of the E(k,p) interval spectra for various values of k and p.

The second method for determining the E(k,p) interval spectrum is a variation on the first which simplifies the determination of an analytic expression in the form of the ratio of two cosine polynomials. This second method is satisfactory for small values of k and p.

III.A.1. The Computational Form Of The E(k,p) Interval Spectrum

The interval spectrum of a stationary point process is defined to be the Fourier transform of the auto-correlation coefficients. That is (2.A.3.3)

$$f_+(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \rho_j \, e^{-ij\omega} \qquad (3.A.1.1)$$

where i is the square root of -1, and

$$\rho_j = \frac{E\{[X_t - E(X)][X_{t+j} - E(X)]\}}{Var(X)} , \quad j = \pm 1, \pm 2, \ldots$$

For the asynchronous process, let

$$N(t) = (\text{the number of events in } (0,t]),$$

and denote the generating function of this asynchronous counting process by

$$\phi(z,t) = \sum_{n=0}^{\infty} z^n \Pr\{N(t) = n\}. \quad 0 \leq z \leq 1. \quad (3.A.1.2)$$

Denote the Laplace transform of $\phi(z,t)$ by

$$\phi^*(z,s) = \int_0^\infty e^{-st}\, \phi(z,t)\, dt. \qquad (3.A.1.3)$$

Since $|\phi(z,t)| \leq 1$ for all z and $t \geq 0$, this transform exists at least for $Re|s| > 0$. Cox and Lewis (1966, Ch. 4) have shown that (3.A.1.1) can be expressed in terms of (3.A.1.3) by (2.A.3.5)

$$f_+(\omega) = \frac{\phi^*(e^{i\omega},0+) + \phi^*(e^{-i\omega},0+)}{\{2\phi\ (0,0+) - E(X)\}} . \qquad (3.A.1.4)$$

Lewis, et al (1973) noted that in the case of the superposition of renewal processes, the Laplace transform of a component asynchronous counting process is easily obtained from the Laplace transform of the interval distribution. In particular, using the identity for the synchronous process (2.A.1.4)

$$N_f(t) \geq n \Longleftrightarrow \sum_{j=1}^{n} X_j \leq t , \qquad (3.A.1.5)$$

the generating function of the synchronous counting process is

$$\phi_f(z,t) = \sum_{n=0}^{\infty} z^n \{F_n(t) - F_{n+1}(t)\} , \qquad (3.A.1.6)$$

where $F_n(t) = Pr\{\sum_{j=1}^{n} X_j \leq t\}$, and $F_o(t) = 1$. Then the

Laplace transform of (3.A.1.6) is

$$\phi_f^*(z,s) = \frac{\{1 - f^*(s)\}}{s\{1 - zf^*(s)\}} , \qquad (3.A.1.7)$$

where $f^*(s) = \int_0^\infty e^{st}\, dF(t)$.

Cox and Lewis (1966, Ch. 4) provide the relationship between the generating functions of the synchronous and asynchronous processes:

$$\phi(z,t) = 1 + \frac{(z-1)}{E(X)} \int_0^t \phi_f(x,u)\, du . \qquad (3.A.1.8)$$

The Laplace transform of (3.A.1.8) is

$$\phi^*(z,s) = \frac{1}{s} + \frac{(z-1)\{1-f^*(s)\}}{s^2 E(X)\ \{1-zf^*(s)\}}$$

$$= \frac{sE(X)\{1-zf^*(s)\} + (z-1)\{1-f^*(s)\}}{s^2E(X)\{1-zf^*(s)\}} . \qquad (3.A.1.9)$$

Thus $\phi^*(z,s)$ for one of the (identically distributed) component processes is a function of the Laplace transform of the interval density. Note further that the generating function of the pooled asynchronous process is, since the component processes are assumed to be independent,

$$\phi^{(p)}(z,t) = E[z^{N_1(t)+\ldots+N_p(t)}]$$

$$= \{E[z^{N_1(t)}]\}^p$$

$$= \phi^p(z,t) \qquad (3.A.1.10)$$

The information required for determining the Laplace transform of the asynchronous superposed process is now available and can, in principle, be used in the following procedure:

$$\phi^*(z,s) \xrightarrow{L^{-1}} \phi(z,t) \xrightarrow{\times} \phi^p(z,t) \xrightarrow{L} \phi^{*(p)}(z,s) , \qquad (3.A.1.11)$$

where L and L^{-1} denote the Laplace transformation and its inverse, respectively.

Example III.A.1. *The E(2,2) process.* We now determine the Laplace transform of the generating function of the asynchronous counts in the E(2,2) process. The Laplace transform of the Erlang density function is

$$f^*(s) = (1 + s/\lambda)^{-2} \qquad (3.A.1.12)$$

and

$$E(X) = 2/\lambda . \qquad (3.A.1.13)$$

Substituting (3.A.1.12) and (3.A.1.13) in (3.A.1.9) leaves

$$\phi^*(z,s) = \frac{2s + (3+z)\lambda}{2\{\lambda^2(1-z) + 2\lambda s + s^2\}} \quad . \qquad (3.A.1.14)$$

To invert (3.A.1.14), it is convenient to substitute $a = -\lambda(1+z)$ and $b = -\lambda(1-z)$, so that

$$\phi^*(z,s) = \frac{2s + (3+z)}{2(s-a)(s-b)} \quad . \qquad (3.A.1.15)$$

Use of a standard inversion formula (Gardner and Barnes, 1942; Selby, 1969) results in

$$\phi(z,t) = (b^2e^{at} - a^2e^{bt})/(4\lambda\sqrt{2}) \quad . \qquad (3.A.1.16)$$

The generating function of the superposed process is then

$$\phi^{(2)}(z,t) = \phi^2(z,t)$$

$$= \{b^4e^{2at} - 2\lambda^4(1-z)^2e^{-2\lambda t} + a^4e^{2bt}\}/(16\lambda^2 z). \qquad (3.A.1.17)$$

The Laplace transform of (3.A.1.17) is

$$\phi^{*(2)}(z,s) = \frac{\frac{b^4}{s-2a} - \frac{2\lambda^4(1-z)^2}{s+2} + \frac{a^4}{s-2b}}{-16\lambda^2 z}. \qquad (3.A.1.18)$$

To continue the general case, Lewis, et al (1973) give the general form for the generating function of asynchronous counts in an Erlang renewal process:

$$\phi(z,t) = (z-1)^2/(k^2 z) \sum_{j=1}^{k} \{\tau_j/(\tau_j-1)^2\}\exp\{-\lambda(\tau_j-1)t\} \qquad (3.A.1.19)$$

where $\tau_1,\ldots,\tau_k$ are the k distinct roots of z. Using (3.A.1.19) to form the corresponding generating function for the superposed process results in

$$\phi^{(p)}(z,t) = \left(\frac{(z-1)^2}{k^2 z}\right)^p \sum_{j_1=1}^{k} \cdots \sum_{j_p=1}^{k} \left(\prod_{j\varepsilon J} \frac{\tau_i}{(\tau_j-1)^2}\right) e^{-\lambda \sum_{j\varepsilon J}(\tau_j-1)t} \qquad (3.A.1.20)$$

where $J = (j_1,j_2,\ldots,j_p)$ is an index vector whose components run from 1 to k. The Laplace transform of (3.A.1.20) is

$$\phi^{*(p)}(z,s) = \left(\frac{(z-1)^2}{k^2 z}\right)^p \sum_{j_1=1}^{k} \cdots \sum_{j_p=1}^{k} \frac{\prod_{j\varepsilon J} \frac{\tau_j}{(\tau_j-1)}}{(s-\lambda)\sum_{j\varepsilon J}(\tau_j-1)} . \qquad (3.A.1.21)$$

Further, it can be shown that by letting

$$Q_\lambda(k,p) = \{[2\phi^{*(p)}(0,0+)-k/(p\lambda)]\}^{-1}$$

that

$$f_+^{(p)}(\omega) = Q_\lambda(k,p)\{(1-\cos\omega)^p/(\lambda k^{2p})\} \sum_{j_1=1}^{k} \cdots \sum_{j_p=1}^{k} \frac{2a_J}{(a_J^2+b_J^2)c_J} \quad , \tag{3.A.1.22}$$

where

$$a_J = p - \sum_{j\varepsilon J} \cos\left(\frac{2\pi j+\omega}{k}\right) \quad ,$$

$$b_J = \sum_{j\varepsilon J} \sin\left(\frac{2\pi j+\omega}{k}\right) \quad ,$$

and

$$c_J = \prod_{j\varepsilon J} \{1-\cos\left(\frac{2\pi j+\omega}{k}\right)\} \quad . \tag{3.A.1.23}$$

This computational form was used by Lewis, et al (1973) to generate numerical representations for several E(k,p) processes, and for k = 2, p = 2, 3 and 4 analytic expressions were derived. It will receive more attention in Chapter VIII below with regard to computation of interval spectra numerically.

It should be noted that $Q_\lambda(k,p)$ (3.A.1.22) is difficult to evaluate directly. Two methods of determining $Q_\lambda(k,p)$ are discussed in Chapter VIII.

III.A.2. A Variation Of The Computational Method For Determining E(k,p) Interval Spectra

The computational form (3.A.1.22) has proven very efficient in the numerical representation of E(k,p) interval

spectra. The modification proposed in this subsection is designed to assist in deriving an analytic expression for the second order interval spectrum of an E(k,p) process. In particular, the relationship of an E(k,p) second order interval spectrum and the spectrum of an ARMA process is explored.

An ARMA(m,n) process has a spectral representation which is the ratio of two finite polynomials in cosine ω. That is, for the ARMA (m,n) process,

$$f_+(\omega) = \frac{\sum_{j=0}^{n} \alpha_j \cos^j \omega}{\sum_{r=0}^{m} \beta_r \cos^r \omega} \quad , \qquad (3.A.2.1)$$

where the limits of summation correspond to the orders of the autoregressive and moving average components of the ARMA process.

It will be shown (Theorem III.A.2; Chapters V and VI) that the E(k,p) second order interval spectrum has the form of (3.A.2.1), hence second order serial properties identical with those of an ARMA process.

This subsection contains the framework through which upper bounds for the parameters m and n of (3.A.2.1) may be established.

Observing that a_J, b_J and c_J (3.A.1.23) depend only on indices in J and not on the order, (3.A.1.22) can be

written in the alternative form

$$f_+^{(p)}(\omega) = Q_\lambda(k,p)\frac{(1-\cos\omega)^p}{\lambda k^{2p}} \sum_{n=1}^{N} \binom{p}{L_n} \frac{2a_{L_n}}{(a_{L_n}^2 + b_{L_n}^2)c_{L_n}} , \tag{3.A.2.2}$$

where $L_n = (r_1,\ldots,r_k)$ is a vector of dimension k corresponding to each distinct vector J which indicates the number of elements of each denomination in J. (The elements r_j, of L_n should retain the subscript n to indicate their relationship to L_n. This form of notation rapidly becomes awkward and where there is no ambiguity the subscripts will be suppressed.) For example, let $k = 5$ and $J = (3,5,3,3)$. The corresponding index designation vector is $L = (0,0,3,0,1)$. We will call L_n an <u>index designation vector</u>. The notation $\binom{p}{L_n}$ is taken to be the multinomial coefficient $p!/(r_1!\ldots r_k!)$, and represents the number of distinct permutations of J. It should also be noted that $\sum_{j=1}^{k} r_j = p$, and that $N = \binom{k+p-1}{p}$ is the figurate number for k types of objects taken in sets of size p (see Riordan, 1958, pp. 25). In this notation, further suppressing the subscript n, the terms of (3.A.1.23) may be written

$$a_L = p - \sum_{j=1}^{k} r_j \cos\left(\frac{2\pi j+\omega}{k}\right) ,$$

$$b_L = \sum_{j=1}^{k} \sin\left(\frac{2\pi j+\omega}{k}\right) ,$$

and

$$c_L = \prod_{j=1}^{k} \{1-\cos\left(\frac{2\pi j+\omega}{k}\right)\}^{r_j} . \qquad (3.A.2.3)$$

Clearly, the coefficients $\binom{p}{L_n}$ do not depend on the order of the elements in L_n. It will be convenient to combine in a single set, R_t, $t = 1,\ldots,T$ those vectors, L_n, which are made up of the same elements. Thus we take $R_t = \{(r_1,\ldots,r_k)\}$ to be the set of distinct k-vectors with elements $\{r_1,\ldots,r_k\}$ in any order. $L \in R_t$ will refer to a particular index designation set. In the above example, $L_{to} = (0,0,3,0,1)$ and $L_{t1} = (3,1,0,0,0)$ would both be members of the same set, $R_t = \{(3,1,0,0,0)\}$. We now restate (3.A.2.2) as

$$f_+^{(p)}(\omega) = Q_\lambda(k,p)\,\frac{(1-\cos\,\omega)^p}{\lambda k^{2p}} \sum_{t=1}^{T} \binom{p}{R_t} \sum_{L\in R_t} \frac{2a_L}{(a_L^2+b_L^2)c_L} . \qquad (3.A.2.4)$$

In this form, $f_+^{(p)}(\omega)$ can be examined in detail by examining the groups of similar terms, i.e., terms with index designators, L, all taken from the same set, R_t.

In the following pages the form of the right hand side of (3.A.2.4) will be examined in detail. It will be shown in Lemmas III.A.1 and III.A.2 that the right hand

side is expressable as the ratio of finite polynomials in cosine(ω/k), with the term $(1-\cos\omega)^p$ cancelling out of the expression. The proofs are constructive and the algebra is cumbersome. However the exposition is considered necessary for the computational work which will be addressed in Chapter VIII.

Lemma III.A.1: For some integer k greater than 1, let R_1 be the set of k-tuples whose elements are distinct permutations of $\{(p,0,0,\ldots,0)\}$. Denote the elements of R_1 by L_n, $n = 1,\ldots,k$. Then

$$\sum_{n=1}^{k} \frac{2a_{L_n}}{(a^2_{L_n}+b^2_{L_n})c_{L_n}} = \frac{(2^{k-1})^p \sum_{j=0}^{p(k-1)} d_j \cos(n\omega/k)}{p(1-\cos\omega)^p} \qquad (3.A.2.5)$$

for some real sequence $\{d_j\}$.

Proof: From (3.A.2.3), and denoting by n the subscript L_n, we have $a_n = p\{1-\cos(\frac{2\pi n+\omega}{k})\}$, $b_n = p\sin(\frac{2\pi n+\omega}{k})$ and $c_n = \{1-\cos(\frac{2\pi n+\omega}{k})\}^p$, $n = 1,\ldots,k$. Evaluating a typical term,

$$a_n^2 + b_n^2 = p^2\{1-2\cos(\tfrac{2\pi n+\omega}{k}) + \cos^2(\tfrac{2\pi n+\omega}{k}) + \sin^2(\tfrac{2\pi n+\omega}{k})\}$$

$$= 2p^2\{1-\cos(\tfrac{2\pi n+\omega}{k})\} = 2pa_n, \quad n = 1,\ldots,k. \qquad (3.A.2.6)$$

Using (3.A.2.6) gives

$$\frac{2a_n}{(a_n^2+b_n^2)c_n} = \frac{1}{pc_n} \qquad (3.A.2.7)$$

and (3.A.2.5) becomes

$$\sum_{n=1}^{k} \frac{2a_n}{(a_n^2+b_n^2)c_n} = \frac{1}{p} \sum_{n=1}^{k} \frac{1}{c_n} \; . \qquad (3.A.2.8)$$

A common denominator for (3.A.2.8) is

$$\overline{D} = p \prod_{n=1}^{k} c_n = p\{ \prod_{n=1}^{k} [1-\cos(\frac{2\pi n+\omega}{k})]\}^p .$$

Jolley (1961, p. 200) gives $\prod_{n=1}^{k} \{1-\cos(\frac{2\pi n+\omega}{k})\} = 2^{1-k}(1-\cos\ \omega)$, with proof credited to E.P. Adams of the Smithsonian Institute. Thus

$$\overline{D} = p\{2^{1-k}(1-\cos\ \omega)\}^p . \qquad (3.A.2.9)$$

Given the common denominator, $\overline{D}$, the numerator in (3.A.2.8) is

$$\overline{N} = \sum_{j=1}^{k} \prod_{n\neq j} \{1-\cos(\frac{2\pi n+\omega}{k})\}^p \; . \qquad (3.A.2.10)$$

Using the standard trigonometric identities

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) ,$$

and

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) , \qquad (3.A.2.11)$$

it is easily shown that

$$\cos\left(\frac{2\pi n+\omega}{k}\right) = \cos(\omega/k)\cos(2\pi n/k) - \sin(\omega/k)\sin(2\pi n/k),$$

$$\cos(2\pi n/k) = \cos(2\pi(k-n)/k),$$

and

$$\sin(2\pi n/k) = -\sin(2\pi(k-n)/k). \qquad (3.A.2.12)$$

Using (3.A.2.12), (3.A.2.10) may be expressed as

$$\begin{aligned}\bar{N} = \sum_{n=1}^{k^*} \{&[1-\cos(\omega/k)\cos(2\pi n/k) + \sin(\omega/k)\sin(2\pi n/k)]^p \\ &+ [1-\cos(\omega/k)\cos(2\pi n/k) - \sin(\omega/k)\sin(2\pi n/k)]^p \} \\ &\times\{ \sum_{j\neq n} [(1-\cos(\omega/k)\cos(2\pi n/k))^2 - (\sin(\omega/k)\sin(2\pi n/k))^2]\}^p \\ &+ \{ \prod_{j=1}^{k^*} [(1-\cos(\omega/k)\cos(2\pi n/k))^2 - (\sin\omega/k)\sin(2\pi n/k))^2]\}U(k)\end{aligned} \qquad (3.A.2.13)$$

where $k^* = \lfloor (k-1)/k \rfloor$, and

$$U(k) = \begin{cases} \{1+\cos(\omega/k)\}^p + \{1-\cos(\omega/k)\}^p, & k \text{ even} \\ \{1-\cos(\omega/k)\}^p, & k \text{ odd.} \end{cases}$$

The symbol $\lfloor x \rfloor$ is taken to mean the largest integer less than or equal to x.

The product terms in (3.A.2.13) have components

$$(1-\cos(2\pi n/k)\cos(\omega/k))^2 - (\sin(2\pi n/k)\sin(\omega/k))^2$$

$$= 1 - \sin^2(2\pi n/k) - 2\cos(\omega/k)\cos(2\pi n/k) + \cos^2(\omega/k)$$

$$= \{\cos(2\pi n/k) - \cos(\omega/k)\}^2, \quad n = 1,\ldots,k^* \qquad (3.A.2.14)$$

Let $A_n = 1 - \cos(2\pi n/k)\cos(\omega/k)$ and $B_n = \sin(2\pi n/k)\sin(\omega/k)$. Then components of the first term of (3.A.2.13) can be written

$$(A_n+B_n)^p + (A_n-B_n)^p = \sum_{j=0}^{p} \binom{p}{j} \{A_n^{p-j}[B_n^j + (-B_n)^j]\}$$

$$= 2 \sum_{j=0}^{p^*} \binom{p}{2j} A_n^{p-2j} B_n^{2j}, \qquad (3.A.2.15)$$

where $p^* = \lfloor (p+1)/2 \rfloor$. Since $B_n = \sin(2\pi n/k)\sin(\omega/k)$, $B_n^{2j} = \{\sin^2(2\pi n/k)[(1-\cos^2(\omega/k)]\}^j$. Using the results of

(3.A.2.14) and (3.A.2.15), (3.A.2.13) can be written

$$\bar{N} = 2 \sum_{n=1}^{k^*} \{ \sum_{j=0}^{p^*} \binom{p}{2j} [1-\cos(\omega/k)\cos(2\pi n/k)]^{p-2j}$$

$$x[\sin(2\pi n/k)]^{2j} [1-\cos^2(\omega/k)]^{j} \}$$

$$x\{ \prod_{j\neq n} [\cos(2\pi n/k) - \cos(\omega/k)]^{2p} \}$$

$$+ \{ \prod_{n=1}^{k^*} [\cos(2\pi n/k) - \cos(\omega/k)]^{2p} \} U(k) \qquad (3.A.2.15a)$$

where

$$U(k) = \begin{cases} \sum_{j=0}^{p^*} \binom{p}{2j} [\cos(\omega/k)]^{2j}, & k \text{ even} \\ [1-\cos(\omega/k)]^{p}, & k \text{ odd} \end{cases}$$

Examination of (3.A.2.15a) reveals that it is a polynomial in $\cos(\omega/k)$. The degree of the polynomial is most easily determined from inspection of (3.A.2.8). Writing

$$\frac{1}{p} \sum_{n=1}^{k} \frac{1}{c_n} = \frac{\sum_{n=1}^{k} \prod_{j\neq n} c_j}{p \prod_{n=1}^{k} c_n},$$

it is clear that each term of the numerator has at most degree (k-1) times the degree of c_n. Since each term

$$c_n = \{(1-\cos(2\pi n/k)\cos(\omega/k) + \sin(2\pi n/k)\sin(\omega/k)\}^p,$$

it is of degree p. Thus there exists a set of coefficients $\{d_n\}$ such that, using standard trigonometric identities (Appendix B)

$$\bar{N} = \sum_{n=0}^{p(k-1)} d_n \cos^n(\omega/k),$$

thus completing the proof of Lemma III.A.1.

It is worth noting that the expression p(k-1) represents a weak upper bound on the degree of the numerator term, $\bar{N}$. A tighter upper bound on the degree is $k\lfloor p(k-1)/k \rfloor$, which follows from Theorem III.A.2 below. This fact is important in the comparison of the E(k,p) interval spectrum with that of a mixed autoregressive/moving average process.

Having established in Lemma II.A.1 the form of a special class of terms of (3.A.2.2), the more general form is considered below. Lemma III.A.2 will show that all terms of the E(k,p) interval spectrum can be expressed in terms of $\cos(\omega/k)$.

Lemma III.A.2: Let R_t be the set of all distinct permutations of the k-tuple $(r_1, r_2, \ldots, r_k)$, with $r_j \geq 0$, and

the sum of the r_i equal to p. Denote the elements of R_t by L_n, n = 1,...,N, where N is determined from specific knowledge of R_t. Then

$$\sum_{n=1}^{N} \frac{2a_{L_n}}{(a_{L_n}^2 + b_{L_n}^2)c_{L_n}} = \frac{\sum_{j=0}^{W_t} d_j \cos \frac{j\omega}{k}}{\sum_{m=0}^{N_t} s_j \cos \frac{j\omega}{k} \left(\frac{1-\cos \omega}{2^{k-1}}\right)^r} \qquad (3.A.2.16)$$

where $r = \max_j\{r_j\}$, and $W_t = N_t + kr - p$.

Proof: It will be shown first that in constructing a common denominator for (3.A.2.16), all terms containing odd powers of $\sin(\omega/k)$ cancel, while even power terms have a cosine representation. Following this, it will be shown that the same phenomena occur in the numerator when the expression is placed over a common denominator.

Examine the denominator of (3.A.2.16) first. The terms c_L, $L \in R_t$, are of the form $\prod_{n=1}^{k} \{1-\cos(\frac{2\pi n+\omega}{k})\}^{r_n}$, and have as a common multiple, $\prod_{n=1}^{k} [1-\cos(\frac{2\pi n+\omega}{k})]^r$, where r is the largest of the r_j, j = 1,...,k. Recalling the result stated in (3.A.2.9), write

$$\bar{D}_c = \{2^{1-k}(1-\cos \omega)\}^r \qquad (3.A.2.17)$$

which coincides with one term in the right hand side of (3.A.2.16).

For the index designation vector $L = (r_1,\ldots,r_k)$, applying the identities (3.A.2.11) gives

$$a_L = p - \cos(\omega/k) \sum_{j=1}^{k} r_j \cos(2\pi j/k) + \sin(\omega/k) \sum_{j=1}^{k} r_j \sin(2\pi j/k),$$

and

$$b_L = \cos(\omega/k) \sum_{j=1}^{k} r_j \sin(2\pi j/k) + \sin(\omega/k) \sum_{j=1}^{k} r_j \cos(2\pi j/k). \tag{3.A.2.18}$$

Using (3.A.2.18), form

$$\begin{aligned} a_L^2 + b_L^2 &= p^2 + \sum_{j=1}^{k} r_j^2 + 2 \sum_{n=1}^{k-1} \sum_{j=n+1}^{k} r_n r_j \cos(2(j-n)\pi/k) \\ &\quad - 2p\{\cos(\omega/k) \sum_{j=1}^{k} r_j \cos(2\pi j/k) - \sin(\omega/k) \sum_{j=1}^{k} r_j \sin(2\pi j/k)\} \\ &= V_L - 2p\{\cos(\omega/k) \sum_{j=1}^{k} r_j \cos(2\pi j/k) - \sin(\omega/k) \sum_{j=1}^{k} r_j \sin(2\pi j/k)\} \end{aligned} \tag{3.A.2.19}$$

with V_L suitably defined.

Now, let $r_i' = r_{k-i}$, $i = 1,\ldots,k-1$, and let $r_k' = r_k$. Then the vector $L' = (r_1',\ldots,r_k')$ will be referred to as the counter element of L in R_t. By applying the identities (3.A.2.11), the following expression is realized:

$$a_{L'}^2 + b_{L'}^2 = V_L$$

$$-2p\{\cos(\omega/k) \sum_{j=1}^{k} r_j \cos(2\pi j/k) + \sin(\omega/k) \sum_{j=1}^{k} r_j \sin(2\pi j/k)\} \qquad (3.A.2.20)$$

which differs from (3.A.2.19) only in the sign of the final summation.

For each element L of R_t there may be at most one counter element L' so formed. Where such a counter element is formed, a common multiple of the denominator in terms of (3.A.2.16) will include the product of (3.A.2.17) and (3.A.2.18). This product is

$$\begin{aligned} & V_L^2 - 4p^2\{\sum_{j=1}^{k} r_j \sin(2\pi j/k)\}^2 \\ & - 4pV_L\{\sum_{j=1}^{k} r_j \cos(2\pi j/k)\}\cos(\omega/k) \\ & + 4p^2\cos^2(\omega/k)\{[\sum_{j=1}^{k} r_j \cos(2\pi j/k)]^2 \\ & + [\sum_{j=1}^{k} r_j \sin(2\pi j/k)]^2\} \end{aligned} \qquad (3.A.2.21)$$

which is a second degree polynomial in $\cos(\omega/k)$.

It is possible that some elements of R_t will not have counter elements. This will occur when, for

$L = (r_1,\ldots,r_k)$, $r_i = r_{(k-i)}$, $i = 1,\ldots,k-1$. In this case

$$\sum_{j=1}^{k} r_j \sin(2\pi j/k) = 0.$$

Thus no $\sin(\omega/k)$ terms will appear in a_L or $(a_L^2+b_L^2)$. Hence the denominator can be expressed as the product of N_2 second degree polynomials in $\cos(\omega/k)$ and $N_1 = N - 2N_2$ first degree polynomials in $\cos(\omega/k)$, along with $\bar{D}_c$ (3.A.2.17). Here, N_2 is the number of element pairs in R_t which are mutually counter elements. As noted after Lemma III.A.1, N is a weak upper bound on the degree of the denominator. A more restrictive upper bound actually realized in some test cases is $N^* = k\lfloor N/k \rfloor$.

Having established the form of the denominator in (3.A.2.16), we continue with the proof of Lemma III.A.2 by examining the numerator. Again, let L and L' be counter elements of R_t. Placing just these terms over a common denominator gives the expression

$$\bar{N}_L = 2a_L(a_L^2,+b_L^2,) \prod_{j=1}^{k} \{1-\cos(\tfrac{2\pi j+\omega}{k})\}^{r-r'_j}$$

$$+ 2a_{L'}(a_L^2+b_L^2) \prod_{j=1}^{k} \{1-\cos(\tfrac{2\pi j+\omega}{k})\}^{r-r_j} \qquad (3.A.2.22)$$

where r is the maximum element of L.

Again, imposing the identities (3.A.2.11) and (3.A.2.12), the product terms of (3.A.2.22) are

$$\prod_{j=1}^{k} \{1-\cos(\tfrac{2\pi j+\omega}{k})\}^{z_j'}$$

$$= \prod_{j=1}^{k} \{1-\cos(\omega/k)\cos(2\pi j/k) + \sin(\omega/k)\sin(2\pi j/k)\}^{z_j'}$$

$$= \prod_{j=1}^{k} \{ \sum_{m=0}^{z_j'} [1-\cos(2\pi j/k)\cos(\omega/k)]^{z_j'-m} [\sin(2\pi j/k)\sin(\omega/k)]^{m} \binom{z_j'}{m} \},$$

(3.A.2.23)

and

$$\prod_{j=1}^{k} \{1-\cos(\tfrac{2\pi j+\omega}{k})\}^{z_j}$$

$$= \prod_{j=1}^{k} \{ \sum_{m=0}^{z_j} [-\cos(2\pi j/k)\cos(\omega/k)]^{z_j-m} [-\sin(2\pi j/k)\sin(\omega/k)]^{m} \binom{z_j}{m} \},$$

(3.a.2.24)

where $z_j = r - r_j$. Clearly, (3.A.2.23) differs from (3.A.2.24) only in the sign of the coefficient of the odd powers of $\sin(\omega/k)$.

Continuing with the proof of Lemma III.A.2, to simplify the exposition, in (3.A.2.18), (3.A.2.19) and (3.A.2.20) make suitable substitutions to let

$$a_L = p - A + B\sin(\omega/k)$$

$$a_L' = p - A - B\sin(\omega/k)$$

$$(a_L^2+b_L^2) = \{V_L - A + B\sin(\omega/k)\}2p$$

$$(a_{L'}^2+b_{L'}^2) = \{V_L - A - B\sin(\omega/k)\}2p$$

so that in (3.A.2.22)

$$a_L(a_{L'}^2+b_{L'}^2) = 2p[V_Lp-pA-AV_L+A^2-B^2\sin^2(\omega/k) + B(V_L-p)\sin(\omega/k)] \qquad (3.A.2.25)$$

and

$$a_{L'}(a_L^2+b_L^2) = 2p(V_Lp-pA-AV_L+A^2-B^2\sin^2(\omega/k) - B(V_L-p)\sin(\omega/k)) \qquad (3.A.2.26)$$

Again, (3.A.2.26) differs from (3.A.2.25) only in the sign coefficient of $\sin(\omega/k)$.

Similarly, the product of (3.A.2.24) and (3.A.2.26) differs from the product of (3.A.2.23) and (3.A.2.25) only in odd powers of $\sin(\omega/k)$, hence summing these two products will result in the cancellation of all odd powers of $\sin(\omega/k)$, leaving only the even powers which are readily convertible to even powers of $\cos(\omega/k)$.

Thus, each pair of terms corresponding to mutually counter elements of R_t, when placed over a common denominator, form a numerator which is a polynomial in $\cos(\omega/k)$. To put

the entire expression (3.A.2.16) over a common denominator, it remains only to multiply $\bar{N}_L$ in (3.A.2.22) by a term of the type shown in (3.A.2.21) for each remaining pair of mutually counter elements in R_t, and a term of the type shown in (3.A.2.20) with the coefficient of $\sin(\omega/k)$ set to zero for each remaining element of R_t. This will retain the form of a polynomial in $\cos(\omega/k)$, thus establishing the nature of the numerator in (3.A.2.16). It remains in the proof of Lemma III.A.2 to derive the degree of the numerator as a polynomial in $\cos(\omega/k)$.

The degree of the polynomial $\bar{N}_L$ (3.A.2.22) when placed over a common denominator is typical, and will have degree

$$W = 2 + 2(N_2-1) + N_1 + \sum_{i=1}^{k} (r-r_i),$$

where N_2 is the number of pairs of mutually counter elements, N_1 is the number of elements without counters and $\sum_{i=1}^{k} (r-r_i)$ is the number of terms which must appear in the numerator to include $\bar{D}_c$ (3.A.2.17) in the denominator. Since $N_1 = N - 2N_2$, $W = N + \sum_{i=1}^{k} (r-r_i)$ and the proof of Lemma III.A.2 is complete.

As before, W actually represents a weak upper bound, with a more realistic bound being $W^* = k\lfloor W/k \rfloor$.

An important implication of Lemmas III.A.1 and III.A.2 is

Theorem III.A.1: The second order interval spectrum of the E(k,p) process may be written

$$f_+^{(p)}(\omega) = Q_\lambda(k,p) \frac{\sum_{n=0}^{N} \alpha_n \cos^n(\omega/k)}{\sum_{m=0}^{M} \beta_m \cos^m(\omega/k)} . \qquad (3.A.2.27)$$

The proof of Theorem III.A.1 is a trivial application of Lemmas III.A.1 and III.A.2. Its value lies in that the interval spectrum of the E(k,p) process can be written in the form of the ratio of finite degree polynomials in $\cos(\omega/k)$. Computational experience (discussed in detail in Chapter VIII) leads to the following:

Theorem III.A.2: Let $f_+^{(p)}(\omega)$ be the second order interval spectrum of an E(k,p) process expressed as in (3.A.2.27). Then the right hand side of (3.A.2.27) can be reduced to

$$f_+^{(p)}(\omega) = Q_\lambda(k,p) \frac{\sum_{n=0}^{N'} \alpha_n' \cos^n\omega}{\sum_{m=0}^{M'} \beta_m' \cos^m\omega} \qquad (3.A.2.28)$$

where $N' = \lfloor N/k \rfloor$, and $M' = \lfloor M/k \rfloor$.

Discussion: The difference between (3.A.2.27) and (3.A.2.28) is that the former has polynomials in $\cos(\omega/k)$ while the latter has polynomials in $\cos\omega$. The significance of this theorem is that it shows the spectrum of intervals

of an Erlang superposition to be the same as that of a mixed autoregressive/moving average process. This permits much of the analysis described by Box and Jenkins (1970) to be applied to analysis of these superpositions.

Even though the intervals of the superposition process are distinctly non-normal, the second order properties upon which much time series analysis is based, are the same for both processes.

Proof of Theorem III.A.2 will be deferred to Chapter VI, where it is shown to be a special case of Theorem V.B.3. Theorem V.B.3 ascribes the form of (3.A.2.29) to the interval spectrum of a semi-Markov generated point process, while Chapter VI deals with modelling the E(k,p) process as such a process.

It might be conjectured that all spectra which can be expressed in the form of (3.A.2.27), which is not the ratio of rational functions of ω, could be further reduced to the form of (3.A.2.28). That this conjecture is particular to the processes considered here and not true in general is shown by the following example:

Let $f_+(\omega) = \sum_{j=0}^{\infty} 2^{-j} \cos j\omega$; $Q_\lambda(k,p) = 1$, $k = 2$, $N = 1$ and $M = 3$. Let the coefficients in the numerator of (3.A.2.27) be 8 and -9. Let the coefficients of the denominator be 10, -9, -8 and 6. The assertion of Theorem III.A.2 is that the coefficients of $\cos(\omega/2)$ and $\cos(3\omega/2)$ must be zero in (3.A.2.27). Some simple calculations will verify that (3.A.2.27) represents an identity in this case.

III.B. SPECIAL CASE RESULTS: E(2,p) AND E(4,p) INTERVAL SPECTRA

The E(2,p) and E(4,p) processes are particularly amenable to analysis because of the simple form of the expansions

$$\cos\left(\frac{2\pi j+\omega}{2}\right) = (-1)^j \cos(\omega/2), \quad j = 1,2,... \tag{3.B.1}$$

$$\sin\left(\frac{2\pi j+\omega}{2}\right) = (-1)^{j+1} \sin(\omega/2), \quad j = 1,2,... \tag{3.b.2}$$

$$\cos\left(\frac{2\pi j+\omega}{4}\right) = \begin{cases} (-1)^j \sin(\omega/4), & j = 1,3,5,... \\ (-1)^{j/2} \cos(\omega/4), & j = 2,4,6,... \end{cases} \tag{3.B.3}$$

$$\sin\left(\frac{2\pi j+\omega}{4}\right) = \begin{cases} (-1)^{(j-1)/2} \cos(\omega/4), & j=1,3,5,... \\ (-1)^{j/2} \sin(\omega/4), & j=2,4,6,... \end{cases} \tag{3.B.4}$$

In the discussion which follows, terms of the form $\cos(n\omega)$ and $(\cos\ \omega)^n$ will be treated interchangeably, except where the exact coefficient of a term is significant to the discussion. This is justified by the trigonometric identities and the Chebyshev polynomials, both summarized in Appendix B, which relate the two forms.

III.B.1. The E(2,p) Interval Spectrum

A tediuous but straightforward construction of the E(2,p) interval spectrum leads to the following

Theorem III.B.1: The second order spectral density of the E(2,p) interval process can be written as the ratio of two polynomials in cos ω as in (3.A.2.28), where $N' = p-1$ and $m' = \lfloor (p-1)/2 \rfloor$.

Proof: For a given value of p, the class, $\mathcal{Q}$, of index designator sets is

$$\mathcal{Q} = \{(p,0),(p-1,1),\ldots,(p-\lfloor p/2 \rfloor, \lfloor p/2 \rfloor)\}. \qquad (3.B.1.1)$$

From Lemma III.A.1, for L = (p,0), and L = (0,p), in the notation of (3.A.2.3),

$$\frac{2a_L}{(a_L^2+b_L^2)c_L} + \frac{2a_{L'}}{(a_{L'}^2+b_{L'}^2)c_{L'}} = \frac{2^p \sum_{j=0}^{p} d_j \cos(j\omega/2)}{p(1-\cos\omega)^p} \qquad (3.B.1.2)$$

which will be denoted by P_o.

Looking more closely at the numerator of (3.B.1.2),

$$\bar{N}_o = c_{} + c_{'} = \{1-\cos(\omega/2)\}^p + \{1+\cos(\omega/2)\}^p$$

$$= 2 \sum_{j=0}^{\lfloor p/2 \rfloor} \binom{p}{2j} \cos^{2j}(\omega/2)$$

$$= 2 \sum_{j=0}^{\lfloor p/2 \rfloor} \binom{p}{2j} (1+\cos\omega)^j, \qquad (3.B.1.3)$$

Using appropriate trigonometric identities and making the obvious substitutions, (3.B.1.3) can be written

$$\bar{N}_o = \sum_{j=o}^{\lfloor p/2 \rfloor} d_j \cos j\omega. \qquad (3.B.1.4)$$

A second obvious case arises when p is even, and $L = (p-(p/2),p/2) = (p/2,p/2)$. Then from (3.A.2.3), (3.B.1.1) and (3.B.1.2), $a_L = p$, $b_L = 0$ and $c_L = 1$, giving

$$2a_L/(a_L^2+b_L^2) = 2/p, \qquad (3.B.1.5)$$

which makes no contribution to the hypothesized ratio of polynomials. When it is defined, the expression (3.B.1.5) will be denoted $P_{p/2}$.

The general term in the E(2,p) spectrum is $R_n = \{(p-n,n),(n,p-n)\}$. In this case, with $L = (p-n,n)$ and $L' = (n,p-n)$, we examine the expression

$$P_n = \frac{2a_L}{(a_L^2+b_L^2)c_L} + \frac{2a_{L'}}{(a_{L'}^2+b_{L'}^2)c_{L'}}, \qquad (3.B.1.6)$$

with a_L, b_L and c_L as in (3.A.2.3).

Using (3.B.1.1) and (3.B.1.2) gives

$$a_L = p + (p-2n)\cos(\omega/2),$$

$$a_{L'} = p - (p-2n)\cos(\omega/2), \qquad (3.B.1.7)$$

so that

$$(a_L^2+b_L^2) = 2\{p^2-2pn+2n^2+(p-2n)\cos(\omega/2)\},$$

$$(a_{L'}^2+b_{L'}^2) = 2\{p^2-2pn+2n^2-(p-2n)\cos(\omega/2)\}, \qquad (3.B.1.8)$$

$$c_L = (1+\cos(\omega/2))^n (1-\cos(\omega/2))^{p-n},$$

and

$$c_{L'} = (1+\cos(\omega/2))^{p-n} (1-\cos(\omega/2))^n. \qquad (3.B.1.9)$$

The common denominator in (3.B.1.6) becomes

$$\bar{D}_n = \{A_n^2-B_n^2\cos^2(\omega/2)\}\{1-\cos^2(\omega/2)\}^{p-n}$$

$$= \{2A_n^2-B_n^2-B_n^2\cos\omega\}\{(1-\cos\omega)/2\}^{p-n}/2, \qquad (3.B.1.10)$$

using standard trigonometric identities, substituting $A_n = p^2-2pn+2n^2$ and $B_n = p-2n$, and recalling from (3.B.1.1) and (3.B.1.5) that n ranges from 1 to $\lfloor(p-1)/2\rfloor$.

The numerator of (3.B.1.6), with (3.B.1.10) as a common denominator, is

$$\bar{N}_n = (p+B_n\cos(\omega/2))(A_n-B_n\cos(\omega/2))(1+\cos(\omega/2))^{p-2n}$$

$$+ (p-B_n\cos(\omega/2))(A_n+B_n\cos(\omega/2))(1-\cos(\omega/2))^{p-2n}$$

$$\bar{N}_n = \{p+A_n+B_n(A_n-p)\cos(\omega/2)-B_n^2\cos^2(\omega/2)$$

$$\times \sum_{j=0}^{p-2n} \binom{p-2n}{j} \cos^j(\omega/2)$$

$$+ \{p+A_n-B_n(A_n-p)\cos(\omega/2)-B_n^2(\cos^2(\omega/2)\}$$

$$\times \sum_{j=0}^{p-2n} \binom{p-2n}{j} (-\cos\ \omega/2)^j$$

$$= 2\{p+A_n-B_n^2\cos^2(\omega/2) \sum_{j=0}^{\lfloor\frac{p-2n}{2}\rfloor} \binom{p-2n}{2j} \cos^{2j}(\omega/2)$$

$$+ 2\{B_n(A_n-p)\} \sum_{j=1}^{\lfloor\frac{p-2n}{2}\rfloor} \binom{p-2n}{2j-1} \cos^2(\omega/2)\,. \qquad (3.B.1.11)$$

Again making simplifying substitutions and invoking the appropriate trigonometric identities, (3.B.1.11) can be expressed as

$$\bar{N}_n = \sum_{j=0}^{\lfloor\frac{p-2n}{2}\rfloor} d_j \cos^j \omega. \qquad (3.B.1.12)$$

Combining (3.B.1.10) and (3.B.1.12) in (3.B.1.6) gives $P_n = \bar{N}_n/\bar{D}_n$. The next step is to evaluate the summation

$$\sum_{n=0}^{\lfloor\frac{p-1}{2}\rfloor} P_n = \frac{1}{(1-\cos\ \omega)^p} \sum_{n=0}^{\lfloor\frac{p-1}{2}\rfloor} \frac{2(1-\cos\ \omega)^n\, \bar{N}_n}{\{2A_n^2-B_n^2(1+\cos\ \omega)\}}. \qquad (3.B.1.13)$$

Recall that in (3.A.2.3) there is a factor $(1-\cos\omega)^p$ so that

$$f_+^{(p)}(\omega) = Q_\lambda(2,p)\frac{(1-\cos\omega)^p}{\lambda 4^p}\sum_{n=0}^{\left\lfloor\frac{p-1}{2}\right\rfloor} P_n$$

$$= \frac{Q_\lambda(2,p)}{\lambda 4^p}\sum_{n=0}^{\left\lfloor\frac{p-1}{2}\right\rfloor}\frac{2(1-\cos\omega)^n \bar{N}_n}{\{2A_n^2-B_n^2(1+\cos\omega)\}} . \qquad (3.B.1.14)$$

Assume p is even. Then each numerator term, $(1-\cos\omega)^n \bar{N}_n$ is a polynomial in cos ω of order (p/2), n = 0,1,...,p/2-1. When the (p/2-1) denominator terms are combined into a common denominator, the term $\bar{N}_o$ is multiplied by a polynomial of order (p/2-1), resulting in a polynomial of order (p-1), which is the maximum degree of any of the numerator terms. Thus for p even, N' and M' of the theorem are confirmed.

Let p be odd. In this case each numerator term $(1-\cos\omega)^n \bar{N}_n$ is a polynomial in cos ω of degree ((p-1)/2), n = 0,1,...,(p-1)/2. The product of $\bar{N}_o$ with the product of the (p-1)/2 denominator terms results in a polynomial of degree (p-1), which is the maximum of all the numerator terms, and again N' and M' are confirmed, thus completing the proof of Theorem III.B.1.

Some specific examples were computed by Lewis et al (1973) and the results are shown in Fig. VIII.A.11.

III.B.2. The E(4,p) Spectrum

The construction of the interval spectrum of the E(4,p) process is also tediuous, but has value in that a

property common to all Erlang superpositions is illustrated. That is, if L^0 and L^1 are members of the same index designator set the terms a_{L1} and a_{L0} may have significantly different form. For example, let k = 6 and p = 3. Take L^0 = (2,1,0,0,0,0)and L^1=(2,0,0,1,0,0) so that, as in (3.A.2.3)

$$a_{L0} = p-2\cos\frac{(2\pi+\omega)}{6} - \cos\frac{(4\pi+\omega)}{6}$$

$$= p-\cos(\omega/6)\{2\cos(\omega/3)+\cos(2\omega/3)\}$$

$$+ \sin(\omega/6)\{2\sin(\omega/3)+\sin(2\omega/3)\} \qquad (3.B.2.1)$$

and

$$a_{L1} = p-2\cos\frac{(2\pi+\omega)}{6} - \cos\frac{(8\pi+\omega)}{6}$$

$$= p-\cos(\omega/6)\{2\cos(\omega/3)+\cos(4\omega/3)\}$$

$$+ \sin(\omega/6)\{2\sin(\omega/3)+\cos(4\omega/3)\}$$

$$= p-\cos(\omega/6)\cos(\omega/3)+\sin(\omega/6)\sin(\omega/3)\,. \qquad (3.B.2.2)$$

The difference between (3.B.1.1) and (3.B.1.2) evolves from the identity

$$\cos((2\pi j+n)/2n) = -\cos(2\pi j/2n). \qquad (3.B.2.3)$$

To construct the E(4,p) interval spectrum, it is necessary to identify those terms, a_L, which are distinct. Comparison of (3.A.2.3) and (3.B.3) reveals that each a_L can be represented as

$$a_L = p + n_1\sin(\omega/4) + n_2\cos(\omega/4) \ , \qquad (3.B.2.4)$$

where $-p \le n_1, n_2 \le p$ and $|n_1| + |n_2| \le p$. A typical set of index designators, R_t, contains the permutations of the non-negative integers $\{l_1,l_2,l_3,l_4\}$. From these permutations are derived the terms

$$\begin{aligned}
a_{Lj} &= p\pm(l_3-l_1)\sin(\omega/4)\pm(l_4-l_2)\cos(\omega/4), \quad j = 1,\ldots,4,\\
a_{Lj} &= p\pm(l_4-l_2)\sin(\omega/4)\pm(l_3-l_1)\cos(\omega/4), \quad j = 5,\ldots,8,\\
a_{Lj} &= p\pm(l_4-l_1)\sin(\omega/4)\pm(l_3-l_2)\cos(\omega/4), \quad j = 9,\ldots,12,\\
a_{Lj} &= p\pm(l_3-l_2)\sin(\omega/4)\pm(l_4-l_1)\cos(\omega/4), \quad j = 13,\ldots,15,\\
a_{Lj} &= p\pm(l_2-l_1)\sin(\omega/4)\pm(l_4-l_3)\cos(\omega/4), \quad j = 17,\ldots,20,\\
a_{Lj} &= p\pm(l_4-l_3)\sin(\omega/4)\pm(l_2-l_1)\cos(\omega/4), \quad j = 21,\ldots,24.
\end{aligned} \qquad (3.B.2.5)$$

Using (3.B.1.5) is inadequate for the analysis as can be seen by looking at two distinct sets of index designation vectors $R_1 = \{1,2,3,4\}$ and $R_2 = \{2,2,3,3\}$. Denoting $L_j \in R_1$ by 1_j and $L_j \in R_2$ by 2_j we have

$$\{a_{1_j}\}_{j=17}^{24} = \{a_{2_j}\}_{j=1}^{8} = \{a_{2_j}\}_{j=9}^{16} .$$

This redundancy must be considered when deriving a common denominator for (3.A.2.4).

To relieve this difficulty define the set $S(p) = \{S_{n,m}\}$, $m = 1,2,\ldots,\lfloor(p/2)-n\rfloor$, $n = 1,2,\ldots,\lfloor p/2\rfloor$, where

$$S_{n,m} = (p-2m-n,n). \qquad (3.B.2.6)$$

Tabulating $S(p)$ gives

$$\begin{aligned} S(p) = \{&(p,0),(p-2,0),\ldots,(p-2\lfloor p/2\rfloor,0),\\ &(p-1,1),(p-3,.),\ldots,((p-2\lfloor(p/2-1\rfloor-1,1),\\ &\vdots\\ &(p-\lfloor p/2\rfloor,\lfloor p/2\rfloor)\}. \end{aligned} \qquad (3.B.2.7)$$

Each element corresponds to a set of ordered pairs $\{(\pm n_1,\pm n_2),(\pm n_2,\pm n_1)\}$ as shown in (3.B.2.4) and (3.B.2.5), and each pair (n_1,n_2) has exactly one corresponding element of $S(p)$, i.e., $(|n_1|,|n_2|)$ or $(|n_2|,|n_1|)$.

The special notation developed here for the E(4,p) superposition process can be used advantageously to prove the following

Theorem III.B.2: The second order spectral density of the E(4,p) process can be written as the ratio of two polynomials in cos ω as in (3.A.2.29), where

$$N' = \begin{cases} (p^2+5p-6)/4, & p = 2,5,6,9,10,... \\ (p^2+5p-4)/4, & p = 3,4,7,8,11,12,... \end{cases} \qquad (3.B.2.8)$$

$$M' = \begin{cases} (p^2+2p-4)/4, & p = 2,4,6,... \\ (p^2+2p-3)/4, & p = 3,5,7,... \end{cases} \qquad (3.B.2.9)$$

Proof: Corresponding to each a_L of (3.A.2.4) is the term

$$(a_L^2+b_L^2) = p^2+n_1^2+n_2^2 + 2p(n_1\sin(\omega/4)+n_2\cos(\omega/4)). \qquad (3.B.2.10)$$

The exact form of c_L (3.A.2.3) is not available from the pair (n_1,n_2). This problem is easily handled in the computation of the denominator since the contribution of all the c_L terms is $p(2^{1-k}(1-\cos\omega))^p$ as stated in (3.A.2.8). The problem in the numerator is more complex and will be solved indirectly following the construction of the denominator.

It is convenient to partition the set $S(p)$ into subsets. Let

$$S^1(p) = \{(p,0)\}$$

$$S^2(p) = \{(p-2j,0), \quad j = 1,\ldots,\lfloor(p-1)/2\rfloor\}$$

$$S^3(p) = \{(j,j), \quad j = 1,\ldots,p/2\}, \ p \text{ even}$$

$$S^4(p) = \{(0,0)\}, \ p \text{ even}$$

and

$$S^5(p) = S(p) - \bigcup_{v=1}^{4} S^v(p) \ . \qquad (3.B.2.11)$$

First, a denominator will be developed for the terms corresponding to particular elements of each of the subsets listed in (3.B.2.11). The denominator corresponding to $S^1(p)$ is given by (3.A.2.9). $S^4(p)$ will also have the

denominator given by (3.A.2.9) with p in that expression replaced by $j = p/2-1, p/2-2, \ldots, p/2-\lfloor p/4 \rfloor$, inclusive, hence making no contribution to the denominator.

A typical term of $S^2(p)$ is $(n_1,0)$, which from (3.B.2.10) and (3.A.2.3), and using standard trigonometric identities, has corresponding common denominator

$$\begin{aligned}\bar{D}_2 &= \{p^2+n_1^2+2pn_1\sin(\omega/4)\}\{p^2+n_1^2-2pn_1\sin(\omega/4)\} \\ &\quad \times\{p^2+n_1^2+2pn_1\cos(\omega/4)\}\{p^2+n_1^2-2pn_1\cos(\omega/4)\} \\ &= (p^2+n_1^2)^4+\{(p^2+n_1^2)2pn_1\}^2+2p^4n_1^4-4p^4n_1^4\cos\,\omega\,. \qquad (3.B.2.12)\end{aligned}$$

A typical element of $S^3(p)$ will yield the common denominator

$$\begin{aligned}\bar{D}_3 &= \{p^2+2n_1^2+2pn_1(\sin(\omega/4)+\cos(\omega/4)\} \\ &\quad \times\{p^2+2n_1^2+2pn_1[\sin(\omega/4)-\cos(\omega/4)]\} \\ &\quad \times\{p^2+2n_1^2-2pn_1[\sin(\omega/4)-\cos(\omega/4]\} \\ &= (p^2+2n_1^2)^4-\{2(p^2+2n_1^2)pn_1\}^2+8p^4n_1^4(1+\cos\,\omega), \qquad (3.B.2.13)\end{aligned}$$

again derived from standard trigonometric identities.

Finally, a typical element of $S^5(p)$ yields the common denominator

$$\bar{D}_5 = \{p^2+n_1^2+n_2^2+2p[n_1\sin(\omega/4)+n_2\cos(\omega/4)]\}$$

$$x(\{p^2+n_1^2+n_2^2+2p[n_2\sin(\omega/4)+n_1\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2+2p[n_2\sin(\omega/4)-n_1\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2+2p[n_1\sin(\omega/4)-n_2\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2-2p[n_1\sin(\omega/4)-n_2\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2-2p[n_2\sin(\omega/4)-n_1\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2-2p[n_2\sin(\omega/4)+n_1\cos(\omega/4)]\}$$

$$x\{p^2+n_1^2+n_2^2-2p[n_1\sin(\omega/4)+n_2\cos(\omega/4)]\}$$

$$= \{(p^2+n_1^2+n_2^2)^4 - (2p(p^2+n_1^2+n_2^2))^2(n_1^2+n_2^2)$$

$$+ 2p^4(n_1^4+n_2^4)(1-\cos\omega)\}^2$$

$$- 64p^8n_1^4n_2^4(1-\cos\omega)^2 . \qquad (3.B.2.14)$$

The degree of the denominator polynomial, M' in (3.B.2.9), is the sum of the degrees of the common denominators for each element of $S(p)$. The cardinality of $S(p)$ is

$$|S(p)| = \sum_{j=0}^{\lfloor p/2 \rfloor} (j+1)$$

$$= \begin{cases} (p^2+6p+8)/8, & p = 2,4,6,..., \\ \\ (p^2+4p+3)/8, & p = 1,3,5,.... \end{cases} \qquad (3.B.2.15)$$

Similarly,

$$|S^1(p)| = 1$$

$$|S^2(p)| = \lfloor (p-1)/2 \rfloor = \begin{cases} p/2-1, & p = 2,4,6,..., \\ \\ (p-1)/2, & p = 3,5,7,..., \end{cases}$$

$$|S^3(p)| = \begin{cases} p/2, & p = 2,4,6,..., \\ \\ 0, & p = 3,5,7,..., \end{cases}$$

$$|S^4(p)| = \begin{cases} 1, & p = 2,4,6,..., \\ \\ 0, & p = 3,5,7,..., \end{cases}$$

$$|S^5(p)| = |S(p)| - \sum_{v=1}^{4} |S^v(p)|$$

$$= \begin{cases} p(p-2)/8, & p = 2,4,6,\ldots, \\ (p^2-1)/8, & p = 3,5,7,\ldots . \end{cases} \qquad (3.B.2.16)$$

From (3.B.2.12), (3.B.2.13), (3.B.2.14) and (3.B.2.16),

$$M' = |S^2(p)|\{\text{degree}(D_2)\} + |S^3(p)|\{\text{degree}(D_3)\}$$

$$+ |S^5(p)|\{\text{degree}(D_5)\}$$

$$= \begin{cases} (p/2-1)+(p/2)+2p(p-2)/8, & p = 2,4,6,\ldots, \\ (p-1)/2+2(p^2-9)/8, & p = 3,5,6,\ldots, \end{cases}$$

$$= \begin{cases} (p^2+2p-4)/4, & p = 2,4,6,\ldots, \\ (p^2+2p-3)/4, & p = 3,5,7,\ldots, \end{cases}$$

which is (3.B.2.9) of the Theorem.

Continuing with the proof, begin analysis of the numerator with $S^1(p)$. The numerator for this set of terms, from (3.A.2.10), (3.B.3) and (3.B.4) is

$$\bar{N}_1 = (1-\cos^2(\omega/4))^p\{(1-\sin(\omega/4))^p + (1-\sin(\omega/4))^p\}$$

$$+ (1-\sin^2(\omega/4))^p\{(1+\cos(\omega/4))^p + (1-\cos(\omega/4))^p\}$$

$$= 2\sum_{j=0}^{\lfloor p/2\rfloor}\binom{p}{2j}\{(\sin(\omega/4))^{2(p+j)} + (\cos(\omega/4))^{2(p+j)}\} \quad (3.B.2.17)$$

To show inductively that

$$(\cos(\omega/4))^{2n} + (\sin(\omega/4))^{2n} = \sum_{j=0}^{\lfloor p/2\rfloor}\alpha_{nj}(\cos\omega)^j, \quad (3.B.2.18)$$

observe from the binomial expansion that

$$(\cos(\omega/4))^{2n} + (\sin(\omega/4))^{2n}$$

$$= 1 = \sum_{n=0}^{n-1}\binom{n}{j}[(\cos(\omega/4))^{2j} + (\sin(\omega/4))^{2j}] . \quad (3.B.2.19)$$

For n = 1,

$$\cos^2(\omega/4) + \sin^2(\omega/4) = 1,$$

and for n = 2

$$\cos^4(\omega/4)+\sin^4(\omega/4) = 1-2\cos^2(\omega/4)+2\cos^4(\omega/4)$$

$$= (3+\cos\omega)/4.$$

Assume that $(\cos(\omega/4))^{2j} + (\sin(\omega/4))^{2j}$ has the form (3.B.2.18) for $j = 1,\ldots,n$. Then

$$(\cos(\omega/4))^{2n+1} + (\sin(\omega/4))^{2n+1}$$

$$= 1 - \sum_{j=1}^{n} \binom{n+1}{j} \sum_{m=0}^{j/2} \alpha_{jm} (\cos\ \omega)^{m}, \qquad (3.B.2.20)$$

with $\lfloor n/2 \rfloor$ providing the maximum power of cos ω in this expression.

Applying this result to (3.B.2.17) gives

$$\bar{N}_1 = \sum_{j=0}^{W} \alpha_j \cos\ j\omega, \quad W = \begin{cases} 3p/4, & p = 2,4,6,\ldots \\ (3p-1)/4, & p = 3,5,7,\ldots \end{cases} \qquad (3.B.2.21)$$

Placing $\bar{N}_1$ over the combined common denominator of degree M' provides a polynomial of degree

$$W' = W+M' = \begin{cases} (p^2+5p-6)/4, & p = 2,5,6,9,10,13,14,\ldots \\ (p^2+5p-4)/4, & p = 3,4,7,8,11,12,\ldots \end{cases} \qquad (3.B.2.22)$$

To show that N' of (3.B.2.8) is equal to W' of (3.B.2.22), it is necessary to examine the structure of (3.A.2.27) in light of (3.B.2.9) and Theorem III.A.1. That is,

$$f_+(\omega) = \sum_{j=0}^{\infty} \gamma_j \cos j\omega$$

$$= Q_\lambda(4,p) \frac{\sum_{n=0}^{N} \alpha_n \cos(n\omega/4)}{\sum_{m=0}^{M'} \beta'_m \cos m\omega} . \qquad (3.B.2.23)$$

Multiplying the left hand summation in (3.B.2.23) by the denominator of the right hand side gives a series in cos $j\omega$. Since this series is equal to a series in $\cos(j\omega/4)$, it is necessary that the only terms with non-zero coefficients in the numerator of the right hand side of (3.B.2.23) have index j an integer multiple of 4, by the orthogonality of the cosine series. Hence all numerator terms for n not an integer multiple of 4 must cancel. It is necessary, then, to be concerned only with those terms which will not cancel.

In particular, the numerator of a given term of $S^2(p)$ will be

$$\bar{N}_2 = \frac{\{p-n_1\sin(\omega/4)\}(1-\cos\omega)^p \bar{D}_2}{\{p/2+n_1^2+2pn_1\sin(\omega/4)\}c_1} .$$

From (3.A.2.3) and the derivation of (3.A.2.8) it is clear that

$$(1-\cos\omega)^p/c_1 = (1-\sin\omega/4)^{l_1}(1+\cos\omega/4)^{l_2}(1-\sin\omega/4)^{l_3} \times (1-\cos\omega/4)^{l_4}$$

where $l_1 + l_2 + l_3 + l_4 = 3p$. Since it is known that all coefficients must vanish except those indexed with integer multiples of 4, N_1 may be considered a polynomial in cos ω of degree $\lfloor(4+3p)/4\rfloor$. Combining this information with (3.B.2.9) shows that the degree of N_1, when placed over the common denominator for the entire expression, to be $\lfloor(4+3p)/4\rfloor+M'-1$ which is equal to W' in (3.B.2.22). The details of analysis for $S^v(p)$, v = 3,4 and 5 lead to the same result, thus completing the proof of Theorem III.B.2.

It has been established that the second order interval spectrum of an E(4,p) process is that of an ARMA(m,n) process with upper bounds given for m and n. Some computations are tabulated in Fig. VIII.A.11 which reflect these results.

III.C. THE DEGREE OF THE SECOND ORDER INTERVAL SPECTRA

In Section A of this chapter, Theorem III.A.2 stated that the second order interval spectrum of the E(k,p) process can be expressed as the ratio of two finite rational polynomials in cos ω (3.A.2.28). It is of interest to explore the implications of this theorem.

For the case in which k is odd, the degree of the denominator is bounded by

$$M' = \left\lfloor \binom{k+p-1}{p} / k \right\rfloor - 1 \quad . \tag{3.C.1}$$

This can be seen by observing that $\binom{k+p-1}{p}$ represents the total number of distinct terms in (3.A.2.2), and it is known from Lemma III.A.1 that there is no contribution to the denominator from the set $R_1 = \{p,0,0,\ldots,0\}$.

Again for k odd, the numerator has degree bounded by $N' = M'+p+\lfloor -p/k \rfloor$. In (3.A.2.16) it was shown that each set R_t could be expressed as the ratio of polynomials of order N_t+kr-p and N_t in $\cos(\omega/k)$. Theorem III.A.2 guarantees that these polynomials can be rewritten as polynomials in $\cos\omega$ of order $\lfloor (N_t+kr-p)/k \rfloor$ and $\lfloor N_t/k \rfloor$. From the construction of R_t, it can be seen that $N_t = k!/(n_1!\ldots n_k!)$ where n_j represents the number of times j appears in R_t. It is easily shown that N_t is an integral multiple of k for all t as defined in Lemma III.A.1. When (3.A.2.16) is placed over the ensemble common denominator, the numerator becomes a polynomial in $\cos\omega$ of order

$$N' = \lfloor (N_t+kr-p)/k \rfloor + (p-r) - \lfloor N_t/k \rfloor + M'$$

$$= M'+p+\lfloor -p/k \rfloor \qquad (3.C.2)$$

The case in which k is even is more difficult. A simple formula is not, in general, available. The procedure followed in the exposition of Theorem III.B.2 must be followed. In (3.B.2.3) it was noted that for $k = 2n$, $\cos(2\pi(j+n)/k) = -\cos(2\pi j/k)$. Further,

$\sin(2\pi(j+n)/k) = -\sin(2\pi j/k)$. Thus, instead of k distinct terms, there are $n = k/2$ distinct terms, each repeated with opposite sign. For example, let $k = 8$ and $p = 3$. Let $L = (2,0,0,0,1,0,0,0)$. Then a_L, as defined in (3.A.2.3), is

$$a_L = p-2\cos\frac{(2\pi+\omega)}{8} - \cos\frac{(10\pi+\omega)}{8}$$

$$= p-\cos\frac{(2\pi+\omega)}{8} .$$

Define the set $V^n(p)$ to be the set of all non-negative vectors of length n satisfying the following conditions:

1. The elements of each vector are arranged in non-decreasing order.

2. The sum of the elements of each vector plus a non-negative even integer is equal to p.

Each vector v in $V^n(p)$ is analogous to an index designator set R_t. For example, if $(1,1,3) \in V^3(9)$, it also represents the vectors (1,3,1) and (3,1,1). In the notation of (3.A.2.3), these vectors represent 24 values of a_L:

$(1,1,3)$: $9\pm\cos((2\pi+\omega)/6)\pm\cos((4\pi+\omega)/6)\pm 3\cos((6\pi+\omega)/6)$

$(1,3,1)$: $9\pm\cos((2\pi+\omega)/6)\pm 3\cos((4\pi+\omega)/6)\pm\cos((6\pi+\omega)/6)$

$$(3,1,1): \quad 9\pm 3\cos((2\pi+\omega)/6)\pm\cos((4\pi+\omega)/6)\pm\cos((6\pi+\omega)/6)$$

Denote the elements of $V^n(p)$ by v_t, $t = 1,\ldots,|V^n(p)|$. Here, $|A|$ denotes the cardinality of the set A. Then the degree of the denominator of the interval spectrum is no larger than

$$M' = \frac{\sum_{t=1}^{|V^n(p)|} |v_t|-2n}{2n} \qquad (3.C.3)$$

and the degree of the numerator is bounded by

$$N' = M'+\max_t\{|(|v_t|-p)/k +p- |v_t|/k|\} , \qquad (3.C.4)$$

where $|v_t|$ represents the number of distinct values of a_L which are derived from v_t. The difference between (3.C.4) and (3.C.2) results from the fact that $|v_t|$ is not necessarily an integral multiple of k. If $v_t = (1,1,1) \varepsilon V^3(3)$, then $|v_t| = 8$, which is not divisible by 6.

Figure III.C.1 is a table of N' and M' values computed by the method shown in this section for various values of k and p. Algebraic computations, discussed in Chapter VIII have confirmed actual values to be in agreement with the estimates for certain specific cases, but lower values are indicated for others.

p \ k	2	3	4	5	6	7	8
2	0,1*	1,2*	1,2*	2,3*	2,3	3,4	3,4
3	1,2*	2,4*	3,5*	6,8*	6,8	11,13	11,13
4	1,3*	4,6	5,8*	13,16	13,16	29,32	27,30
5	2,4	6,9	8,11	24,28	23,27	65,69	56,60
6	2,5	8,12	11,15	41,45	36,40	131,136	115,122
7	3,6	11,15	15,20	65,70	41,53	245,251	
8	3,7	14,19	19,25	98,104	74,80	489,497	
9	4,8	17,23	24,30	142,149	108,115		
10	4,9	21,27	29,36	199,207	142,150		

Figure III.C.1 Upper bound estimates for the orders of the polynomials representing E(k,p) second-order interval spectra. The first element of each pair is degree of the denominator. The second is the degree of the numerator. Those processes marked with an asterisk have been computed and are shown in Figure VIII.C.5. Serial correlation analysis of the E(3,5) process indicate the actual degree is (6,8) rather than the indicated (6,9). See Section VIII.E.

III.D. COMMENTS ON THE MIXED AUTOREGRESSIVE/MOVING AVERAGE SPECTRUM

It has been stated (Theorem III.A.2) that the second order E(k,p) spectrum has the form of a mixed autoregressive/moving average process of degree m and n respectively. An important property of a stationary mixed autoregressive/moving average process is the relationship of the serial correlations. A typical finite ARMA(m,n) process is defined by

$$z_t+a_1z_{t-1}+\ldots+a_mz_{t-m} = e_t+b_1e_{t-1}+\ldots+b_ne_{t-n} \qquad (3.D.1)$$

where z_t is the observed value of the process at time t and $\{e_k\}$ is a sequence of independent identically distributed random variables with zero mean. With m > n the serial correlations are given by (Box and Jenkins, 1970)

$$\rho_k = a_1\rho_{k-1}+\ldots+a_m\rho_{k-m}, \quad k > n, \qquad (3.D.2)$$

For the first n lags, $\rho_1,\ldots,\rho_n$ depend jointly on a_j, $j = 1,\ldots,m$, and b_i, $i = 1,\ldots,n$.

Since the second order spectrum of a stationary interval process is an unique, invertable transformation of the serial correlations, it follows that the serial correlations of an E(k,p) process have the above property. Thus, those sections of Box and Jenkins dealing with model identification can be applied directly to a data set which is suspected of having emanated from an E(k,p) process. Figure III.C.1

can be used in this respect to assist in determining which E(k,p) process might provide the best model.

Estimation properties for k, p and λ of the E(k,p) process are unknown. As a result, the statistician must apply at once the methodology of Chapter VIII of Cox and Lewis (1966) which bases the estimation of p on the variance-time curve and the additivity of the counting process in a superposition; the results of this chapter regarding interval series analysis, and the results of Chapter VII of this thesis which relate the marginal interval density to the number and nature of the component processes.

III.E. THE SECOND ORDER COUNT SPECTRUM FOR THE E(k,p) PROCESS

In Chapter II, Subsection A.2, the second order spectrum of counts of a stationary point process is introduced following Cox and Lewis (1966, Ch. 4). In subsection II.B.3, the extension to the count spectrum of the superposition of independent stationary point processes was given (Cox and Lewis, 1966, Ch. 8).

For the sake of completeness, a sketch of those results is presented here, and applied to the Erlang superposition.

Recalling (2.A.2.5) and (2.A.2.6), the asynchronous intensity of a stationary (regular) process is $m = 1/E[X]$, where X denotes the length of an interval following an arbitrary event, and the synchronous intensity is $m_f(t)$.

The intensity function of the synchronous counting process generated by the superposition of p independent renewal processes is

$$m_f^{(p)}(\tau) = m_f(\tau) + (p-1)m \quad . \tag{3.E.1}$$

The count spectrum of a stationary stochastic process is

$$g_+(\omega) = m\{1+ \int_{-\infty}^{\infty} (m_f(u)-m)e^{-i\omega u}du\}/\pi\mu, \quad \omega \geq 0$$

which has the alternative form, due to Cox and Lewis

$$g_+(\omega) = m\{1+m_f^*(i\omega)+m_f^*(-i\omega)\}/\pi\mu, \tag{3.E.2}$$

where $m_f^*(s)$ is the ordinary one-sided Laplace transform of $m_f(t)$. From (2.A.2.6), it is clear that for a renewal process

$$m_f^*(s) = \frac{f^*(s)}{\{1-f^*(s)\}} \quad , \tag{3.E.3}$$

where $f(t)$ is the renewal density and $f^*(s)$ is its Laplace transform. Then for the superposition of renewal processes, (3.E.2) can be expressed as

$$g_+^{(p)}(\omega) = \left(\frac{p}{\pi\mu}\right)\left\{\frac{1+f^*(i\omega)f^*(-i\omega)}{1-f^*(i\omega)-f^*(-i\omega)+f^*(i\omega)f^*(-i\omega)}\right\}$$

$$= pg_+(\omega) \qquad (3.E.4)$$

The spectrum of the E(k,p) counting process is given by (3.E.4) with $f^*(s) = (\lambda/(\lambda+s))^k$, so that

$$g_+(\omega) = \left(\frac{p\lambda}{k\pi}\right)\left\{\frac{(\lambda^2+\omega^2)^k - \lambda^{2k}}{(\lambda^2+\omega^2)-\lambda^k\{(\lambda+i\omega)^k+(\lambda-i\omega)^k\}+\lambda^{2k}}\right\}$$

$$= \left(\frac{p\lambda}{k\pi}\right)\frac{\sum_{j=1}^{k}\binom{k}{j}\left(\frac{\omega^2}{\lambda^2}\right)^{j-1}}{\sum_{j=1}^{k}\binom{k}{j}\left(\frac{\omega^2}{\lambda^2}\right)^{j-1} - 2\sum_{j=1}^{k/2}\binom{k}{2j}(-1)^j\left(\frac{\omega^2}{\lambda^2}\right)^{j-1}} \qquad (3.E.5)$$

Evaluating (3.E.5) at 0 gives

$$g_+(0+) = p\lambda/(k^2\pi) \ . \qquad (3.E.6)$$

IV. SECOND ORDER SPECTRAL ANALYSIS OF THE $H(k,p;q,\lambda)$ PROCESS

The analysis of the last chapter dealt with renewal processes with interevent times governed by the Erlang distribution. One important property of the Erlang distribution is that it has increasing hazard rate. That is, if X has increasing hazard rate, then $\frac{d}{dt}\{f(t)/R(t)\}$ is an increasing function of t. See Barlow and Prochan (1965) for a more detailed discussion of the hazard property. Enns (1970) showed that if p iid renewal processes are superposed, and the component processes have monotone hazard rate, then an arbitrary interval of the superposition process has monotone hazard rate.

It is of interest, therefore, in case one observed a superposition of renewal processes in which the times between superposed events had a decreasing hazard rate, to examine a class of distributions with decreasing hazard rate for the intervals of the component renewal process. One such distribution is the hyperexponential which has the form of a convex combination of exponential distributions. The hyperexponential distribution may be thought of as a set of exponentially distributed random variables, the choice of which is governed by a discrete probability law. (A more general definition has the parameter of the exponential distribution selected from an arbitrary probability

distribution on the positive real line. The definition given for this research project is a special case of the more general definition.) The tractability of this distributional form in obtaining an interval spectrum for the superposition process comes from the fact that the Laplace transform of the pdf, as in the Erlang case, is a ratio of rational functions in s.

Section A is devoted to development of the interval spectrum of the hyperexponential superposition process using the method described in Chapter III.

In Section B, the interval spectrum for the $H(2,2;\underline{q},\underline{\lambda})$ is worked out in detail.

The count spectrum of the $H(2,p;\underline{q},\underline{\lambda})$ process is shown in Section C. The complexity of the form of the Laplace transform of the hyperexponential pdf precludes in depth analysis of either spectral form presented in this chapter.

IV.A. THE INTERVAL SPECTRUM OF THE $H(k,p;\underline{q},\underline{\lambda})$ PROCESS

Following the methodology of Chapter III, Section A, the interval spectrum will be determined from the procedure

$$\phi^{*}(z,s) \overset{L}{\leftarrow} \phi(z,t) \leftarrow \phi^{p}(z,t) \overset{L}{\leftarrow} \phi^{*(p)}(z,s), \qquad (4.A.1)$$

where $\phi(z,t)$ is the generating function of counts for the asynchronous component process, and $\phi^{*}(z,s)$ is its ordinary Laplace transform. The symbols L, and L^{-1} denote the

Laplace transformation and the inverse Laplace transformation, respectively.

Let $N_i(t)$ be a component asynchronous counting process. Denote the survivor function of the interevent times in the component renewal processes by

$$R(t) = \sum_{j=1}^{k} q_j \exp\{-\lambda_j t\}, \qquad (4.A.2)$$

where $q_j > 0$, $\lambda_j > 0$, $j = 1,\ldots,k$, and, $\sum_{j=1}^{k} q_j = \sum_{j=1}^{k} \lambda_j = 1$. Then the superposition of the p iid processes is an $H(k,p;\underline{q},\underline{\lambda})$ process.

The Laplace transform of the component interevent pdf is

$$\begin{aligned} f^*(s) &= -\int_0^\infty \frac{dR(t)}{dt} e^{-st}\, dt \\ &= \sum_{j=1}^{k} q_j \lambda_j \exp\{-(\lambda_j+s)t\}dt \\ &= \sum_{j=1}^{k} (q_j\lambda_j)/(s+\lambda_j). \end{aligned} \qquad (4.A.3)$$

From (3.A.1.9)

$$\begin{aligned} \phi^*(z,s) &= \frac{sE[X]\{1-zf^*(s)\}+(z-1)\{1-f^*(s)\}}{s^2E[X]\{1-zf^*(s)\}} \\ &= \frac{s\left(\sum_{j=1}^{k} \frac{q_j}{\lambda_j}\right)\left(1-z\sum_{j=1}^{k} \frac{q_j\lambda_j}{s+\lambda_j}\right)+(z-1)\left(1-\sum_{j=1}^{k} \frac{q_j\lambda_j}{s+\lambda_j}\right)}{s^2\left(\sum_{j=1}^{k} \frac{q_j}{\lambda_j}\right)\left(1-z\sum_{j=1}^{k} \frac{q_j\lambda_j}{s+\lambda_j}\right)}. \end{aligned} \qquad (4.A.4)$$

Now, let

$$U(s) = \prod_{j=1}^{k} (s+\lambda_j)$$

and

$$H(s) = \sum_{j=1}^{k} q_j \lambda_j \prod_{r \neq j} (s+\lambda_r), \qquad (4.A.5)$$

so that (4.A.3) can be rewritten as

$$f^*(s) = H(s)/U(s). \qquad (4.A.6)$$

Using (4.A.6), the Laplace transform of the generating function of counts, (4.A.4) can be written

$$\phi^*(z,s) = \frac{sE[X]\{U(s)-zH(s)\}+(z-1)\{U(s)-H(s)\}}{s^2E[X]\{U(s)-zH(s)\}}$$

$$= A(s)/B(s), \qquad (4.A.7)$$

where A(s) and B(s) are defined in the obvious way.

Then Gardner and Barnes (1942) give

$$L^{-1}(\phi^*(z,s)) = \phi(z,t)$$

$$= \sum_{m=1}^{M} A(s_m)\exp\{s_m t\}/B'(s_m) , \qquad (4.A.8)$$

where the values s_m, m = 1,...,M, are the distinct poles of $\phi^*(z,s)$, provided there are first order poles only. No effort has been made to determine if, in general, $\phi^*(z,s)$ has only first order poles. This has been the case for those specific processes considered. The prime denotes the first derivative with respect to s.

The inversion of $\phi^*(z,s)$ is not a trivial matter, even when provided with the general inversion formula. A more manageable approach is presented in Chapter VI, wherein the $H(k,p;\underline{q},\underline{\lambda})$ process is represented as a semi-Markov generated point process with equivalent probabilistic structure. Further analysis along current lines will, therefore, be restricted to the $H(2,2;\underline{q},\underline{\lambda})$ process.

IV.B. THE INTERVAL SPECTRUM OF THE $H(2,2;\underline{q},\underline{\lambda})$ PROCESS

The component process residual life distribution for time between events is, from (4.A.2)

$$R(x) = q_1 \exp\{-\lambda_1 x\} + q_2 \exp\{-\lambda_2 x\},$$

with corresponding Laplace transform

$$f^*(s) = [s(q_1\lambda_1+q_2\lambda_2)+\lambda_1\lambda_2]/(s^2+s+\lambda_1\lambda_2), \qquad (4.B.1)$$

and expectation

$$E[X] = (q_1\lambda_2+q_2\lambda_1)/(\lambda_1\lambda_2). \qquad (4.B.2)$$

Evaluating the Laplace transform of the count generating function (4.A.7) in light of (4.A.6), (4.B.1) and (4.B.2) requires tedious but straightforward algebra. The result is

$$\phi^*(z,s) = \frac{s+1-z(q_1\lambda_1+q_2\lambda_2)+\lambda_1\lambda_2(z-1)(q_1\lambda_2+q_2\lambda_1)}{s^2+s\{1-z(q_1\lambda_1+q_2\lambda_2)\}+\lambda_1\lambda_2(1-z)}$$

$$= \frac{s-\alpha+\lambda_1\lambda_2(z-1)/(q_1\lambda_2+q_2\lambda_1)}{(s-a)(s-b)} , \qquad (4.B.3)$$

where

$$\alpha = z(q_1\lambda_2+q_2\lambda_1)-1,$$

$$\beta = \alpha^2-4\lambda_1\lambda_2(1-z) ,$$

$$a = (\alpha+\beta)/2 ,$$

$$b = (\alpha-\beta)/2 . \qquad (4.B.4)$$

Inverting $\phi^*(z,s)$ using (4.A.6) results in

$$\phi(z,t) = [(2\gamma-\alpha+\beta)\exp\{at\}-(2\gamma-\alpha-\beta)\exp\{bt\}]/2 , \qquad (4.B.5)$$

where

$$\gamma = \lambda_1\lambda_2(z-1)/(q_1\lambda_2+q_2\lambda_1) . \qquad (4.B.6)$$

Continuing with the procedure outlined in (4.A.1),

$$\phi^{2}(z,t) = (2\beta^{2})^{-1}[(2\gamma-\alpha+\beta)^{2}\exp\{2at\}$$

$$+ (2\gamma-\alpha-\beta)^{2}\exp\{2bt\}$$

$$- 8(\gamma^{2}-\gamma\alpha+\lambda_1\lambda_2(1-z)\exp\{\alpha\}] \ , \qquad (4.B.7)$$

with α, β, γ, a and b as in (4.B.4) and (4.B.6).

Since α, β, γ, a and b are independent of t, the Laplace transform of (4.B.7) is immediate:

$$\phi^{*(2)}(z,s) = (2\beta^{2})^{-1}[(2\gamma-\alpha+\beta)^{2}/(s-2a)$$

$$+ (2\gamma-\alpha-\beta)^{2}/(s-2b)$$

$$- 8\{\gamma^{2}-\gamma\alpha+4\lambda_1\lambda_2(1-z)\}/(s-\alpha)] \ . \qquad (4.B.8)$$

Recalling from (3.A.4) that the interval spectral density of a stationary point process is given by

$$f_{+}(\omega) = \frac{\phi^{*}(e^{i\omega},0+)+\phi^{*}(e^{-i\omega},0+)}{(2\phi^{*}(0,0+)-E[X])\pi} \ , \qquad (4.B.9)$$

the next step is to evaluate $\phi^{*(2)}(z,s)$ at $s = 0+$, so that

$$\phi^{*(2)}(z,0+) = -2\{\alpha^2-2\alpha\gamma+\gamma^2+\lambda_1\lambda_2(1-z)\}/\{\alpha(\alpha^2-\beta^2)\}$$

$$= \frac{(z-1)^2(A-\lambda_1\lambda_2)^2+A[z^2(z^2-2A+2\lambda_1\lambda_2)+z(2A-3\lambda_1\lambda_2)+\lambda_1\lambda_2]}{2A^2\lambda_1\lambda_2[(z-1)^2-Az(z-1)]} , \qquad (4.B.10)$$

where

$$A = q_1\lambda_2+q_2\lambda_1 = 1-q_1\lambda_1-q_2\lambda_2 . \qquad (4.B.11)$$

In Equation (4.B.10), let

$$R = A^4-2A^3+A^2(1+2\lambda_1\lambda_2)+2A\lambda_1\lambda_2+(\lambda_1\lambda_2)^2 ,$$

$$S = 2A^3-A^2(2-3\lambda_1\lambda_2)+4A\lambda_1\lambda_2-2(\lambda_1\lambda_2)^2 ,$$

$$T = A^2(1+\lambda_1\lambda_2)-2A\lambda_1\lambda_2-2(\lambda_1\lambda_2)^2 ,$$

$$U = 2A^2\lambda_1\lambda_2(1-A) ,$$

$$V = 2A^2\lambda_1\lambda_2(A-2) ,$$

and

$$W = 2A^2\lambda_1\lambda_2 , \qquad (4.B.12)$$

so that

$$\phi^{*(2)}(z,0+) = \frac{Rz^2+Sz+T}{Uz^2+Vz+W} . \qquad (4.B.13)$$

Now, making the appropriate substitutions for z in (4.B.13), and recalling (4.B.2), (4.B.9) and (4.B.11), the interval spectrum of the superposition process may be written as

$$f_+^{(2)}(\omega) = Q \frac{(RW+UT)\cos 2\omega+\{V(R+T)+(U+W)S\}\cos\omega+RU+SV+WT}{UW\cos 2\omega+V(U+W)\cos\omega+U^2+V^2+W^2} , \qquad (4.B.14)$$

where

$$Q = [A^2\lambda_1\lambda_2]/[2\lambda_1\lambda_2-2A\lambda_1\lambda_2+A^2(1-\lambda_1\lambda_2)-A^3). \qquad (4.B.15)$$

Beyond the observation that the interval spectrum of the H(2,2;$\underline{q}$,$\underline{\lambda}$) process is that of an ARMA(2,2) process, very little insight is gained from this method of spectral computation. Answers to such questions as the nature of the serial correlation sequence as a function of $\underline{q}$ and $\underline{\lambda}$, the effect of increased values of k or p, etc., are deeply burried in the simplifying notation. Further analysis along this line appears futile.

IV.C. THE COUNT SPECTRUM OF THE H(2,p;$\underline{q}$,$\underline{\lambda}$) PROCESS

Using the procedure of Section III.E, the second order spectrum of the counting process is more easily obtained.

For simplicity, attention is restricted to the $H(2,p;\underline{\alpha},\underline{\lambda})$ process.

Equations (3.E.6), (4.B.1) and (4.B.2) yield the following expression for the spectral density of this counting process:

$$g_{+}^{(p)}(\omega) = pA\{A^2-2(1-\lambda_1\lambda_2)+\omega^2\}/\{(A^2+\omega^2)(\lambda_1\lambda_2)\} \ , \quad (4.C.1)$$

where A is as defined in the set of Equations (4.B.11).

V. SEMI-MARKOV GENERATED POINT PROCESSES

A class of processes which are of direct application to this research is that of events marking transitions in a semi-Markov process. These processes are studied in this chapter. Chapter VI is devoted to describing the Erlang and hyperexponential superposition processes as semi-Markov generated point processes, and applying the results of this chapter.

In Section A we give the definition of a semi-Markov process and univariate semi-Markov generated point process, with a brief summary of previous work with regard to this model. Section B contains the development of the interval covariance function and second order interval spectrum of a semi-Markov generated point process. In Section C the results of Section B are expanded to the interval trivariance and bispectrum of intervals, with an indication of the method of extension to higher order multivariance and spectra.

The interest in the higher order joint moments and spectra comes from the fact that the intervals in a semi-Markov generated point process are non-normal; thus the process is not completely specified by second order joint moments. In examining the higher order moments we hope to give a more complete description of the process.

V.A. THE SEMI-MARKOV GENERATED POINT PROCESS DEFINED

A very general definition formulated by Smith (1955) of a semi-Markov process has n^2 independent, non-negative random variables, S_{ij}, $i,j = 1,\ldots,n$, with distribution functions $F_{ij}(v)$, which are sampled in accordance with a Markov chain with transition matrix $T = \{t_{ij}\}$. At a transition time, if the Markov chain makes a transition to state i, the next transition is to j with probability t_{ij} and the time between transitions is distributed in accordance with $F_{ij}(v)$.

If each transition of the Markov chain is recorded as an event in time, the record of events is a semi-Markov generated point process.

In Smith's work, and most subsequent work, the semi-Markov process is either a model for a state process, the sample functions taking on n levels, or a multivariate point process (Cox and Lewis, 1972), i.e., a point process with n event types called a Markov renewal process. Here, it is used as the source of an interval sequence in a univariate point process by superposing the marginal processes. Cherry (1972) developed the mathematical structure of the superposition of two Markov renewal processes as multivariate point processes.

As the model of a multivariate point process, it is apparent that the sequence of times between successive arrivals into state i in a semi-Markov process forms a

renewal process for each state i = 1,...,n. The distinctive feature is that the n renewal processes thus formed are dependent. The semi-Markov generated point process is, then, a superposition of dependent renewal processes. In general, n independent renewal processes form a semi-Markov process only if the intervals are exponentially distributed. It will be shown in Chapter VI, however, that it is possible to construct a semi-Markov generated point process for which the probabilistic structure of the (output) sequence of events is the same as that of the superposition of finitely many Erlang or Hyperexponential independent renewal processes.

For the purpose of this analysis, assume the distributions $F_{ij}(v)$ are absolutely continuous and have an associated density function $f_{ij}(v)$. Using bold face type to indicate matrices, denote by $\mathbf{F}(v) = \{f_{ij}(v)\}$ the matrix of transition time densities. Further, assume the non-central moments $\mu_{ij}^{(r)} = E[S_{ij}^{r}]$ exist for $r \leq R$, and let $\mathbf{E}_r = \{\mu_{ij}^{(r)}\}$ be the matrix of r-th moments.

It is worth noting at this point that in much of what follows, the non-central moments of the random variables $\{S_{ij}\}$ are all that is required, hence the assumption of absolute continuity is unnecessarily strong. The reasons for the supposition are to simplify the exposition, to ensure that the point process will be regular, and finally, that in the applications proposed the assumption is satisfied, thus negating a requirement for a more rigorous treatment.

Let X_m be the random variable associated with the m-th transition of the Markov chain, for $m = 0,\pm1,\pm2,\ldots$. Thus if the m-th transition is from state i to state j, $X_m = S_{ij}$. That is, the residence, or sojourn time in state i, conditioned on eventual transition to state j. Assuming the origin of the sequence $\{X_m\}$ was chosen arbitrarily, and all states of the chain are ergodic (see Feller, 1968, or Kemeny and Snell, 1960), then $\{X_m\}$ is a stationary sequence of intervals. This fact is most easily seen by observing that the transition process is stationary, so that the probability of entering a particular state following the k-th event approaches a constant as k goes to infinity, regardless of the starting conditions. Hence, the finite dimensional distributions of $\{X_m\}$ are translation invariant with respect to the index sequence.

Example V.A.1 *A Two State Univariate Semi-Markov Generated Point Process*. Cox and Lewis (1966, p. 194), propose the following example of a univariate semi-Markov point process:

$$T = \begin{vmatrix} \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & \alpha_2 \end{vmatrix}, \quad 0 \leq \alpha_1, \alpha_2 \leq 1,$$

$$F(t) = \begin{vmatrix} f_1(t) & f_1(t) \\ f_2(t) & f_2(t) \end{vmatrix}, \quad t \geq 0.$$

This model might describe the failure pattern for a machine which receives its spare parts from two sources, each source having its own failure time distribution. When $\alpha_1 = \alpha_2 = 0$, an alternating renewal process is generated.

V.A.1. The Second Order Count Spectrum of a Semi-Markov Generated Point Process

Rudemo (1973) analyzed in detail a special case of a semi-Markov generated point process. In the notation of this chapter,

$$f_{ij}(v) = \begin{cases} \lambda_{ij}\exp\{-\lambda_{ij}v\}, & i \neq j \\ 0 , & i = j, \end{cases} \tag{5.A.1.1}$$

and

$$t_{ij} = \begin{cases} \lambda_{ij} / \sum_{i \neq j} \lambda_{ij} , & i \neq j \\ 0 , & i = j. \end{cases} \tag{5.A.1.2}$$

In addition, define the matrix, Q, as the matrix of elements

$$q_{ij} = \begin{cases} \lambda_{ij} , & i \neq j, \\ -\sum_{i \neq j} \lambda_{ij} , & i = j. \end{cases} \tag{5.A.1.3}$$

The interpretation of q_{ij} is that

$$q_{ij} = \lim_{dt \to 0} \{P_{ij}(dt) - P_{ij}(0)\}/dt$$

where $P_{ij}(t)$ = Pr{system is in state j at t|state i at 0}. That is, q_{ij} is the intensity of a transition to state j given the system is in state i.

Assuming the matrix T is irreducible and aperiodic, it has stationary vector, $\underline{\pi}$, such that

$$\underline{\pi}^T T = \underline{\pi}^T \quad . \tag{5.A.1.4}$$

Similarly, define p_j, the j-th element of $\underline{p}$, to be the limit as time goes to infinity of the probability that the system is in state j; i.e., the probability that the system is in state j at an arbitrary time. Then,

$$p_j = \{q_{jj}(\sum_n (1/q_{nn})\}^{-1} \quad . \tag{5.A.1.5}$$

Let the matrix $P(t) = \{p_{ij}(t)\}$. Then as t goes to infinity, $P_{ij}(t) \to p_j$, and $P(\infty)$ is a matrix for which each row is $\underline{p}^T$. Also, $P'(0) = Q$, and

$$p'_{ij}(t) = \sum_n P_{in}(t) q_{nj} \quad , \tag{5.A.1.6}$$

where the "prime" is used to denote the first derivative.

Finally, Rudemo has determined that the asynchronous intensity of an event at an arbitrary time is

$$m = \underline{p}^{\top}\underline{q} = \underline{\pi}^{\top}P(\infty)\underline{q} \ , \qquad (5.A.1.7)$$

and the asynchronous intensity is

$$m_f(t) = \underline{\pi}^{\top}P(t)\underline{q} \ , \qquad (5.A.1.8)$$

where the vector $\underline{q} = \{q_{jj}\}$, and $\underline{\pi}^{\top}$ indicates the transpose of $\underline{\pi}$.

Having defined and characterized the basic model, Rudemo developed a representation for the second order count spectrum of the process. Following Cox and Lewis (1966, Ch. 4), and as given by (2.A.2.13),

$$g_+(\omega) = \frac{m}{\pi}\{1+m_f^*(i\omega)+m_f^*(-i\omega)\}, \quad \omega \geq 0, \qquad (5.A.1.9)$$

where

$$m_f^*(s) = \int_0^{\infty} e^{-st}(m_f(t)) \ dt.$$

From (5.A.1.7) and (5.A.1.8)

$$m_f^*(s) = \int_0^{\infty} e^{-st}\underline{\pi}^{\top}(P(t)-P(\infty))\underline{q} \ dt$$

$$m_f^*(s) = \underline{\pi}^T P^*(s)\underline{q} \ , \qquad (5.A.1.10)$$

where

$$P^*(s) = \int_0^\infty e^{-st}(P(t) - P(\infty))dt.$$

Some simple calculations will verify that $P(\infty)Q = 0$, so from (5.A.1.6)

$$\frac{d}{dt}\{P(t)-P(\infty)\} = \{P(t)-P(\infty)\}Q, \qquad (5.A.1.11)$$

giving

$$P^*(s) = \{I-P(\infty)\}(Is-Q)^{-1}. \qquad (5.A.1.12)$$

Using (5.A.1.7), (5.A.1.10) and (5.A.1.12) in (5.A.1.9) yields

$$g_+(\omega) = \underline{\pi}^T\{P(\infty)+(I-P(\infty))[(I\omega i-Q)^{-1}+(-I\omega i-Q)^{-1}]\}\underline{q}. \qquad (5.A.1.13)$$

Two points of interest with regard to Rudemo's work as related to this thesis are worth noting here. First, Rudemo's definition of a semi-Markov generated point process is actually somewhat more general than given here in that he permits a subset of states for which transitions are not recorded in the point process. This is similar to the

methodology in Chapter VI below wherein some states are deleted from the state space representation in order to reduce the size of the matrices.

Secondly, the computational procedures presented in Subsections VIII.B.1 and VIII.B.2 below for finding the inverse of a matrix (Iu-T) can be applied to (5.A.1.13) if a numerical representation of the second order count spectrum is desired.

V.B. SECOND ORDER INTERVAL PROPERTIES OF THE SEMI-MARKOV GENERATED POINT PROCESS

Let T be an nxn stochastic matrix with stationary vector as in (5.A.1.4). Each transition is recorded according to its time of occurrence, with times between transitions governed by F(u), the matrix of transition time densities. Let X_j represent the interval following the j-th event, with an arbitrary interval denoted either X or X_o, and with S_{ik} the time to transition from state i to state k in a single step. Let $E_r = \{\mu_{ij}^{(r)}\}$ be the matrix of r-th moments for r = 1,2,3,...,R.

Define two new matrices:

$G(u) = \{t_{ij} f_{ij}(u)\}$ represents a matrix of transition densities weighted by the Markov probability of the transition;

$M_r = \{t_{ij} \mu_{ij}^r\}$ represents a matrix of r-th transition moments weighted by the Markov probability of the transition.

The following results may now be established:

Lemma V.B.1: The arbitrary interval, X, is governed by the pdf

$$f_X(u) = \underline{\pi}^T G(u) \underline{1} \qquad (5.B.1)$$

where $\underline{1}$ represents an n-vector of 1's. Thus $f_X(u)$ is a convex linear combination of the $f_{ij}(u)$'s.

Define T^∞ to be the limit as $j \to \infty$ of T^j. Then we have

Lemma V.B.2:

$$E[X] = \underline{\pi}^T M_1 \underline{1} \ , \qquad (5.B.2)$$

$$Var[X] = \underline{\pi}^T (M_2 - M_1 T^\infty M_1) \underline{1} \ , \qquad (5.B.3)$$

From Lemmas V.B.1 and V.B.2 we have

Theorem V.B.1:

$$Cov(X_t, X_{t+j}) = \underline{\pi}^T M_1 (T^{j-1} - T^\infty) M_1 \underline{1}, \quad j = 1,2,3,\ldots, \qquad (5.B.4)$$

and

Theorem V.B.2: The spectral density of the interval process is

$$f_+(\omega) = \underline{\pi}^T\{M_2+M_1[(Ie^{i\omega}-T)^{-1}+(Ie^{-i\omega}-T)^{-1}]M_1\}\underline{1}/\sigma^2, \qquad (5.B.5)$$

$$0 \leq \omega \leq \pi,$$

where

$$\sigma^2 = \mathrm{Var}[X].$$

Proof of Lemma V.B.1: Let v be the state of the process following an arbitrary event. $\Pr\{v=i\} = \pi_i$, where π_i is the i-th element of $\underline{\pi}$. The probability that the next event marks a transition to state j is t_{ij}. Thus $\Pr\{X_o=S_{ij}\} = \pi_i t_{ij}$ with conditicnal density $f_{ij}(u)$. The unconditional density is

$$f_0(u) = \sum_{i=1}^{n} \Pr(v=i) \sum_{j=1}^{n} t_{ij} f_{ij}(u)$$

$$= \underline{\pi}^T G(u)\underline{1}. \qquad (5.B.6)$$

This completes the proof of Lemma V.B.1.

Proof of Lemma 2: The expectation of X is finite since it may be no larger than max $\{u_{ij}\}$, which is finite by assumption. Thus interchange of order in the formal definition using (5.B.6) yields

$$E[X] = \int_0^\infty u f_o(u)du$$

$$= \sum_{i=1}^{n} \Pr(v=i) \sum_{j=1}^{n} t_{ij} \int_0^\infty u f_{ij}(u)du$$

$$= \sum_{i=1}^{n} \Pr(v=i) \sum_{j=1}^{n} t_{ij}\mu_{ij}$$

$$= \underline{\pi}^T M_1 \underline{1} \ , \qquad (5.B.7)$$

which agrees with (5.B.2).

Similarly,

$$E[X^2] = \underline{\pi}^T M_2 \underline{1} \ , \text{ hence}$$

$$\mathrm{Var}[X_o] = \underline{\pi}^T \{M_2 - M_1(\underline{1}\ \underline{\pi}^T)M_1\}\underline{1} \ . \qquad (5.B.8)$$

Note that $\underline{1}\ \underline{\pi}^T$ is a nxn matrix each row of which is identical with $\underline{\pi}^T$. Feller (1968, Ch. 15) shows that as j tends to infinity T^j tends to $\underline{1}\ \underline{\pi}^T = T^\infty$, under the assumptions of this chapter. Thus

$$\mathrm{Var}[X_o] = \underline{\pi}^T (M_2 - M_1 T^\infty M_1)\underline{1} \ .$$

This completes the proof of Lemma V.B.2.

Proof of Theorem V.B.1: Since $\{X_j\}$ is a stationary sequence, write for $j \geq 1$

$$\mathrm{Cov}(X_t, X_{t+j}) = \gamma(j) = E[X_o X_j] - E[X]^2 .$$

Applying the Chapman-Kolmogorov forward equation (see for example Cox and Miller, 1965) the conditional expectation of $X_o X_j$ is

$$E[X_o X_j | X_o = S_{sk}] = \mu_{sk} \sum_{i=1}^{n} t_{ki}^{(j-1)} \sum_{j=1}^{n} t_{i\ell} \mu_{i\ell} , \qquad (5.B.9)$$

where $t_{ki}^{(p)}$ represents the probability that the state of the system at step p is i given that the state at step 0 was k, i.e., the p-th stage transition probability. Note that $T^p = \{t_{ij}^{(p)}\}$ (Feller, 1968, Ch. 15).

Unconditioning (5.B.9),

$$\begin{aligned} E[X_o X_j] &= \sum_{s=1}^{n} \sum_{k=1}^{n} \Pr(X = S_{sk}) E[X_o X_j | X = S_{sk}] \\ &= \sum_{s=1}^{n} \pi_s \sum_{k=1}^{n} t_{sk} \mu_{sk} \sum_{i=1}^{n} t_{ki}^{(j-1)} \sum_{\ell=1}^{n} t_{i\ell} \mu_{i\ell} \\ &= \underline{\pi}^T M_1 T^{j-1} M_1 \underline{1} . \qquad (5.B.10) \end{aligned}$$

Thus,

$$\begin{aligned} \gamma(j) &= \underline{\pi}^T M_1 T^{j-1} M_1 \underline{1} - (\underline{\pi}^T M_1 \underline{1})^2 \\ &= \underline{\pi}^T M_1 (T^{j-1} - T^{\infty}) M_1 \underline{1} , \qquad (5.B.11) \end{aligned}$$

which corresponds to (5.B.4) of Theorem V.B.1, thus completing the proof of Theorem V.B.1.

Proof of Theorem V.B.2: Where it exists, the interval spectral density is defined by

$$f_+(\omega) = \{1+(2/\sigma^2) \sum_{j=1}^{\infty} \gamma(j)\cos j\omega\}/\pi,$$

$$= \{1+(1/\sigma^2) \sum_{j=1}^{\infty} \underline{\pi}^{\top} M_1 (T^{j-1}-T^{\infty}) M_{1-} (e^{i\omega j}+e^{-i\omega j})\}/\pi,$$

$$0 \leq \omega \leq \pi , \qquad (5.B.12)$$

with $i = \sqrt{-1}$.

Let $C(z) = \sum_{j=1}^{\infty} \gamma(j) z^j$ denote the serial covariance generating function. If $\sum_{j=1}^{\infty} \gamma(j) < \infty$, then $C(z) < \infty$, $0 \leq |z| \leq 1$. We first show that the covariance sequence is summable, and that $\lim_{z \to 1-} C(z) = \sum_{j=1}^{\infty} \gamma(j)$. Then, we show that $(Ie^{i\omega}-T)^{-1}$ exists for $\omega \neq 2j\pi$, $j = 0,\pm 1,\pm 2,\ldots$.

The basis for the summability of the covariance sequence is from Kemeny and Snell (1960, p. 71), where it is shown that there are constants b and r, $0 < r < 1$, such that

$$\left| t_{jk}^{(v)} - \pi_k \right| \leq br^v, \quad j,k = 1,\ldots,n; \quad v = 1,2,\ldots \qquad (5.B.13)$$

Combining (5.B.10), (5.B.11) and (5.B.15) yields

$$|\gamma(j)| = \sum_{s=1}^{n} \pi_s \sum_{k=1}^{n} t_{sk} m_{sk} \sum_{i=1}^{n} |t_{ki}^{(j-1)} - \pi_i| \sum_{\ell=1}^{n} t_{i\ell} m_{i\ell}$$

$$\leq \sum_{s=1}^{n} \pi_s \sum_{k=1}^{n} t_{sk} m_{sk} \sum_{i=1}^{n} b r^{j-1} \sum_{\ell=1}^{n} t_{i\ell} m_{i\ell}$$

$$= [b \sum_{s,k,i,\ell=1}^{n} \cdots \sum \pi_s t_{sk} m_{sk} t_{i\ell} m_{i\ell}] r^{j-1} . \qquad (5.B.14)$$

Clearly, the right hand side of the inequality (5.B.14) is a term of a summable geometric series, hence $\gamma(j)$ is absolutely summable.

From the Lebesque Convergence Theorem (Royden, 1963, p. 200) if $g(x)$ is integrable, $|f_n(x)| \leq g(x)$ and $f_n(x) \to f(x)$ then $\int f(x) = \lim_{n\to\infty} \int f_n(x)$. Identifying $|\gamma(j)| = g(j)$, $f_n(j) = \gamma(n)(\frac{n-1}{n})^j e^{ij/n}$ and $z = (\frac{n-1}{n})^j e^{ij/n}$, we have

$$\sum_{j=1}^{\infty} \gamma(j) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \gamma(j)(\frac{n-1}{n})^j e^{ij/n} = C(1) \qquad (5.B.15)$$

It remains to show that $(Ie^{-i\omega} - T)^{-1}$ exists.

If the determinant $|Ix-T| = 0$ then the inverse $(Ix-T)^{-1}$ is not defined. However, $|Ix-T| = 0$ is the characteristic equation for the Markov chain described by T and there are at most n distinct roots. Since T is ergodic, $x_1 = 1$ is a root, and for all other roots the inequality $|x| < 1$ holds (Feller, 1968, Ch. 16). Since $|e^{-i\omega}| = 1$, and for $\omega \neq 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, $e^{-i\omega} \neq 1$, $e^{-i\omega}$ is not a characteristic value of T for $0 < \omega < 2\pi$, hence $(Ie^{-i\omega} - T)^{-1}$ exists.

Now, let $z = e^{-i\omega}$ for some $0 < \pi < 2\pi$. Then from (5.B.11),

$$C(e^{-i\omega}) = \sum_{j=1}^{\infty} \underline{\pi}^T M_1 (T^{j-1} - T^{\infty}) M_1 \underline{1} e^{-i\omega}$$

$$= \underline{\pi}^T M_1 \{(Ie^{i\omega} - T)^{-1} - T^{\infty}(e^{-i\omega} - 1)^{-1}\} M_1 \underline{1} \quad , \quad (5.B.16)$$

and

$$f_+(\omega) = \{1 + (1/\sigma^2)[C(e^{-i\omega}) + C(e^{i\omega})]\}/\pi$$

$$= \{1 + \underline{\pi}^T M_1 [(Ie^{i\omega} - T)^{-1} + (Ie^{-i\omega} - T)^{-1} + T^{\infty}] M_1 \underline{1}/\sigma^2\}/\pi. \quad (5.B.17)$$

Equation (5.B.15) extends $f_+(\omega)$ to $\omega = 0$ and $\omega = 2\pi$. Using (5.B.4),

$$f_+(\omega) = \underline{\pi}^T \{M_2 + M_1 (Ie^{i\omega} - T)^{-1} + (Ie^{-i\omega} - T)^{-1}] M_1\} \underline{1}/(\sigma^2 \pi),$$

which is Equation (5.B.5), thus completing the proof of Theorem V.B.2.

The form of the spectrum is an important part of this research. In particular, the form of the interval spectrum of a semi-Markov generated point process is that of a mixed moving average-autoregressive process. The following theorem states this fact explicitly:

Theorem V.B.3: The spectral density (5.B.5) can be written as

$$f_+(\omega) = Q \frac{\sum_{j=1}^{J} \alpha_j \cos j}{\sum_{k=0}^{K} \beta_k \cos k} , \qquad (5.B.18)$$

where Q, J, K, α_j, j = 1,...,J and β_k, k = 1,...,K depend on T, M_1, M_2, and on the size of the state space, n.

Proof: Examine the form

$$\underline{\pi}^T M_1 (Ix-T)^{-1} M_1 \underline{1} = Z(x) . \qquad (5.B.19)$$

Assuming x is not a characteristic value of T, the inverse exists, and Z(x) is a scaler. Inverting (Ix-T) by the adjoint method (see for example Finkbeiner, 1966) the adjoint elements will be polynomials in x of degree n-1 or less. The determinant |Ix-T| is a polynomial of degree n, so (5.B.19) can be written

$$Z(x) = \{\underline{\pi}^T M_1 \mathrm{Adj}(Ix-T) M_1 \underline{1}\} / |Ix-T| , \qquad (5.B.20)$$

which is the ratio of (n-1)th and n-th degree polynomials in x, so

$$Z(x) = \frac{\sum_{j=0}^{n-1} a_j x^j}{\sum_{k=0}^{n} b_k x^k} \quad . \tag{5.B.21}$$

Now,

$$Z(e^{i\omega})+Z(e^{-i\omega}) = \frac{2\sum_{j=0}^{n-1}\sum_{k=0}^{n} a_j b_k \cos(j-k)}{2\sum_{k=0}^{n-1}\sum_{\ell=k+1}^{n} b_k b_\ell \cos(k-\ell) + \sum_{j=0}^{n} b_j^2} \quad . \tag{5.B.22}$$

Substituting (5.B.22) in (5.B.5) with suitable identification of terms results in (5.B.22) which is the spectrum of a finite mixed autoregressive-moving average process.

This completes the proof of Theorem V.B.3.

V.C. EXTENSIONS TO HIGHER ORDER SPECTRA

Defining the <u>trivariance</u> function in a natural way, let

$$Trv(X,Y,Z) = E[\{X-E(X)\}\{Y-E(Y)\}\{Z-E(Z)\}].$$

Then for the stationary sequence of synchronous intervals in a semi-Markov generated point process, we have

Theorem V.C.1: $Trv(X_r, X_{r+j}, X_{r+j+k}) = \tau(j,j+k)$

$$= \underline{\pi}^T M_1 [T^{j-1} M_1 T^{k-1} - T^{\infty} M_1 (T^{j-1} + T^{k-1} + T^{j+k-1} - 2T^{\infty})] M_1 \underline{1},$$

$$j,k \geq 1 \tag{5.C.1}$$

Proof: Since the process is stationary,

$E[X_r] = E[X_{r+j}] = \mu$ and

$$\mathrm{Trv}(X_r, X_{r+j}, X_{r+j+k}) = \tau(j, j+k)$$

$$= E[(X_o-\mu)(X_j-\mu)(X_{j+k}-\mu)]$$

$$= E[X_o X_j X_{j+k}]-\mu E[X_o X_j+X_o X_k+X_o X_{j+k}]+2\mu^3. \qquad (5.C.2)$$

Arguing as in the proof of Theorem V.B.1,

$$E[X_o X_j X_{j+k}] = \underline{\pi}^T M_1 T^{j-1} M_1 T^{k-1} M_1 \underline{1}. \qquad (5.C.3)$$

Substituting (5.C.3) and (5.B.10) in (5.C.2) yields (5.C.1).

This completes the proof of Theorem V.C.1.

Similarly,

$$\tau(j,j) = \underline{\pi}^T M_1[(T^{j-1}-T^\infty)M_2-2T^\infty M_1(T^{j-1}-T^\infty)]M_1\underline{1}\ , \qquad (5.C.4)$$

$$\tau(0,j) = \underline{\pi}^T M_2 T^{j-1} M_1 - M_1 T^\infty [2M_1 T^{j-1} M_1 + M_2 - 2M_1 T^\infty M_1]\}\underline{1}, \qquad (5.C.5)$$

and

$$\tau(0,0) = \underline{\pi}^T[M_3 - T^\infty(3M_2-2M_1T^\infty M_1)]\underline{1}\ . \qquad (5.C.6)$$

Rosenblatt and Van Ness (1966) list the following symmetries in the trivariance function of a stationary process:

$$\tau(j,k) = \tau(k,j) = \tau(-j,k-j), \tag{5.C.7}$$

which indicates that $\tau(j,k)$ is completely specified by its values over any one of the following six sectors along with the boundries:

Section I: $0 < j < k$

Section II: $0 < k < j$

Sector III: $j < 0 < k$

Sector IV: $j < k < 0$

Sector V: $k < j < 0$

Sector VI: $k < 0 < j$ (5.C.8)

Define the bispectrum of intervals by

$$b(\omega_1,\omega_2) = \Big[\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\tau(j,k)\exp\{-i\omega_1 j - i\omega_2 k\}\Big]/[4\pi^2\tau(0,0)],$$

$$-\pi \leq \omega_1,\omega_2 \leq \pi. \tag{5.C.9}$$

In view of (5.C.8), (5.C.9) can be expressed in terms of sector I values only:

$$b(\omega_1,\omega_2) = \{ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \tau(j,j+k)[e^{-i((\omega_1+\omega_2)j+\omega_2 k)} + e^{-i((\omega_1+\omega_2)j+\omega_1 k)}$$

$$+ e^{i((\omega_1+\omega_2)k+\omega_2 j)} + e^{i((\omega_1+\omega_2)k+\omega_1 j)} + e^{-i(k\omega_1-j\omega_2)} + e^{i(j\omega_1-k\omega_2)}]$$

$$+ \sum_{j=1}^{\infty} \tau(0,j)[e^{-i\omega_2 j} + e^{-i\omega_1 j} + e^{i(\omega_1+\omega_2)j}] + \tau(0,0)$$

$$+ \sum_{j=1}^{\infty} \tau(j,j)[e^{+i\omega_1 j} + e^{i\omega_2 j} + e^{-i(\omega_1+\omega_2)j}]\}/(4\pi^2\tau(0,0) \quad (5.C.10)$$

To evaluate (5.C.10) in view of (5.C.1), (5.C.5), (5.C.6) and (5.C.7), the following formulae will be useful. For a, b such that e^{-a}, $e^{-b} < 1$ and not characteristic values of T,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T^{j-1} M_1 T^{k-1} e^{-aj-bk} = (Ie^a - T)^{-1} M_1 (Ie^b - T)^{-1}, \quad (5.C.11)$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} e^{-aj-bk} = (e^a - 1)^{-1}(e^b - 1)^{-1}, \quad (5.C.12)$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T^{j-1} e^{-aj-bk} = (e^b - 1)^{-1}(Ie^a - T)^{-1}, \quad (5.C.13)$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} T^{j+k-1} e^{-aj-bk} = T(Ie^a - T)^{-1}(Ie^b - T)^{-1}, \quad (5.C.14)$$

Let $\tau^*(a,b) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \tau(j,k+j)e^{-aj-bk}$. Then from (5.C.1), (5.C.4) through (5.C.7) and (5.C.11) through (5.C.14),

$$\tau^*(a,b) = \pi^T M_1\{(Ie^a-T)^{-1}(Ie^{-b}-T)^{-1}-T^\infty M_1[(e^b-1)^{-1}(Ie^a-T)^{-1}$$

$$+ (e^a-1)^{-1}(Ie^b-T)^{-1}+T(Ie^b-T)^{-1}(Ie^a-T)^{-1}$$

$$- 2T^\infty(e^a-1)^{-1}(e^b-1)^{-1}]\}M_1\underline{1} \cdot \qquad (5.C.15)$$

Let

$$\tau^*(a) = \sum_{j=1}^{\infty} \tau(j,j)e^{-aj}$$

$$= \underline{\pi}^T M_1\{[(Ie^a-T)^{-1}T^\infty(e^a-1)^{-1}]M_2$$

$$- 2TM_1[(e^a-1)^{-1}T^\infty(e^a-1)^{-1}]M_1\}\underline{1} \qquad (3.C.16)$$

and

$$\tau_o^*(a) = \sum_{j=1}^{\infty} \tau(0,j)e^{-aj}$$

$$= \underline{\pi}^T M_2(Ie^a-T)^{-1}M_1\underline{1}-2\underline{\pi}^T M_1 T^\infty(Ie^a-T)^{-1}M_1\underline{1}$$

$$-\underline{\pi}^T\{M_1T^\infty M_2\underline{1}(e^a-1)^{-1}+2M_1T^\infty M_1T^\infty M_1\underline{1}(e^a-1)^{-1}\} \quad (5.C.17)$$

Finally, substituting (5.C.15), (5.C.16) and (5.C.17) in (5.C.10) gives the result

$$b(\omega_1,\omega_2) = \{\tau^*((\omega_1+\omega_2)i,\omega_2 i)+\tau^*((\omega_1+\omega_2)i,\omega_1 i)+\tau^*(-\omega_1 i,-(\omega_1+\omega_2)i)$$

$$+ \tau^*(-\omega_2 i,-(\omega_1+\omega_2)i)+\tau^*(-\omega_2 i,\omega_1 i)+\tau^*(\omega_1 i,-\omega_2 i)$$

$$+ \tau^*((\omega_1+\omega_2)i)+\tau^*(-\omega_1 i)+\tau^*(-\omega_2 i)+\tau(0,0)$$

$$+ \tau_o^*(-(\omega_1+\omega_2)i)+\tau_o^*(\omega_1 i)+\tau_o^*(\omega_2 i)\}/(4\pi^2\tau(0,0)) \qquad (5.C.18)$$

The existence of $b(\omega_1,\omega_2)$ is shown in a manner similar to that in the proof of Theorem V.B.2.

In general, n-th order spectra are defined as Fourier transforms of functions of n-th order joint moments, frequently cumulants (see Brillanger, 1965, 1972; Tukey, 1959). The key step in determining the n-th order spectrum of any moment function is to observe that, that for this process,

$$E[X_o X_j \ldots X_{x+k}]$$

$$= \underline{\pi}^T M_1 (T^{j-1} M_1 \ldots M_1 T^{k-1}) M_1 \underline{1}, \quad j,\ldots,k = 1,2,3,\ldots \qquad (5.C.19)$$

and

$$E[X \ldots X_{v+j}^r \ldots X_{s+k}]$$

$$= \underline{\pi}^T M_1 (\ldots T^{j-1} M_r \ldots T^{k-1}) M_1 \underline{1} \; . \qquad (5.C.20)$$

Thus the joint interval structure is completely characterized in terms of $M_1,\ldots,M_r,\underline{\pi}$, and T, with spectral representation depending on the choice of moment function.

It may be observed at this point that the joint interval distributions are directly accessible. Referring to Lemma V.B.1, we see that the arbitrary interval density is a linear combination of the n^2 transition densities. Combining this idea with the construction of (5.C.19) gives the result

$$f_{o,j,k}(u_u,u_2,u_3) = \pi^T G(u_1)T^{j-1}G(u_2)T^{k-1}G(u_3)\underline{1} , \qquad (5.C.21)$$

where $f_{o,j,k}(u,v,w)$ represents the joint density of the intervals 0,j and j+k. This result can be generalized to any combination of intervals.

VI. THE ERLANG AND HYPEREXPONENTIAL SUPERPOSITION PROCESSES AS SEMI-MARKOV GENERATED POINT PROCESSES

In Chapters III and IV, the nature of the spectral representation of Erlang and hyperexponential superposition processes was examined. The algebraic manipulations proved difficult and unweildy for all but the simplest processes. In this chapter, the "method of stages" is exploited to derive semi-Markov generated point processes which are equivalent with the Erlang and hyperexponential superposition processes.

In addition to the computational savings resulting from the semi-Markov representation, higher order spectral estimates become feasible. The development of higher order spectra is desireable because the various processes are not uniquely defined by second order spectra which are indistinguishable from those of a stationary mixed autoregressive/moving average process. Higher order joint moments are not, in general, available for the general superposition of iid renewal processes, but can be determined for the Erlang and hyperexponential superpositions by means of the semi-Markov modeling described below. In Section A the E(k,p) process is shown to have an equivalent semi-Markov generated point process representation. Independence beyond the first lag is shown for the E(2,2) process, and computational considerations are discussed with regard to determining the matrices

T and E_r of Chapter V for the general E(k,p) process. State space reduction and compression receive special attention.

Section B is devoted to the modeling of the $H(k,p;\underline{q},\underline{\lambda})$ process as a semi-Markov generated point process.

VI.A. THE E(k,p) PROCESS AS A SEMI-MARKOV GENERATED POINT PROCESS

Let $Y = X_1+X_2+\ldots+X_k$, where $Pr[X_j>x] = e^{-\lambda x}$. Then Y has the probability density function

$$f_Y(y) = \lambda^k y^{k-1} e^{-\lambda y}/(k-1)!$$

$$= \gamma_y(k,\lambda) \quad .$$

This well known fact that the sum of k exponentially distributed random variables has the Erlang distribution is the key property which allows the modeling of an Erlang superposition process as a semi-Markov process.

In the "method of stages" the random variable Y is thought of as proceeding through k stages, the delay times independent and with identical exponential distributions over all stages. A realization of Y occurs upon termination of the k-th stage. Y is said to be in stage i at t if

$$X_1+X_2+\ldots+X_i \leq t$$

and

$$X_1+X_2+\ldots+X_{i+1} > t \quad .$$

Assume that p independent renewal processes with identical Erlang inter-event distributions are observed concurrently. A state space can be defined in the following way:

Designate each process as process i, i=1,...,p, and denote the current stage of process i by s_i=1,...,k. Then define the state space

$$S = \{(s_1,\ldots,s_p)\}$$

as the set of all combinations of stages.

A transition occurs whenever a component process advances one stage and each component process cycles repetitively through stages 1,...,k,1,...,k,1,... A renewal event occurs in process i whenever it makes the transition from s_i=k to s_i=1.

Note that in this formulation direct transitions are possible from $\underline{s}^0=(s_1^0,\ldots,s_p^0)$ to $\underline{s}^1=(s_1^1,\ldots,s_p^1)$ only if for some i=1,...,p $(\underline{s}^0+\underline{e}_i)_{\text{mod } k}=\underline{s}^1$, where $\underline{e}_i$ is a vector of zeroes except for a one in the i-th position.

In this particular formulation, transition times are the same for all transitions. Because of the lack of memory property of the exponential distribution, if the system is observed at any time, the time to the next stage in any process has the residual life distribution $R(t) = e^{-\lambda t}$. The next transition is the minimum over the p processes of the time to the next stage. Thus the time to the next

transition in the superposition process has the residual life distribution $R^{(p)}(t) = e^{-p\lambda t}$. Clearly, the process which actually advances is process i with probability 1/p, since we assume that the component processes are identically distributed and independent.

Example VI.A.1. *The E(3,2) process.* Let k=3 be the shape parameter of the Erlang renewal distribution, and let p=2 be the number of component processes. The state space is

$$S=\{(1,1),(1,2),(2,1),(2,2),(3,2),(2,3),(3,3),(3,1),(1,3)\} . \qquad (6.A.1)$$

Figure VI.A.1 shows a network of possible paths of the Markov chain through the state space. When the process enters a state enclosed in a box, a component process event occurs, and the process continues as from the circled state of the same number.

Recalling that the residence time in any state has residual life distribution $R^{(p)}(t) = e^{-p\lambda t}$, observe that if the process is in state (1,1), there are two paths by which the process can leave and reenter that state without any intervening events. Since each branch from the state is chosen with probability 1/2, the probability that a particular path is chosen is 2^{-n}, where n is the length of the path. Thus the lower branch from (11) to [11] will be realized with probability 2^{-3}. Given that a particular path is chosen, the

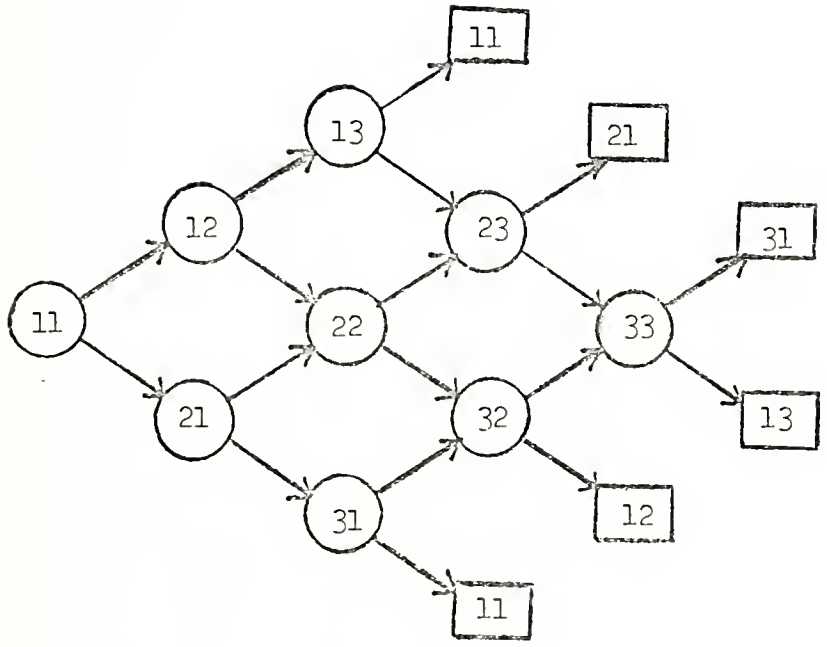


Figure VI.A.1 Transition network for the E(3,2) process. Times of transitions into states enclosed in boxes are recorded as events in the point process. The transition process continues from the circled state with the same index pair.

time until the next event is governed by the density

$$\gamma_t(n,p\lambda) = (p\lambda)^n t^{n-1} e^{-p\lambda t}/(n-1)! \qquad (6.A.2)$$

In this example, if the process is in state (1,1), the probability that the next event is associated with a transition back into state (1,1) is

$$\Pr[(1,1):(1,1)] = 2(1/8) = 1/4 \quad ,$$

while the time until the next event, given that it represents a return to (1,1) is governed by $\gamma_t(3,2\lambda)$ with expectation $3/2\lambda$. The colon is used to denote a direct transition between the states indicated.

We will return to this example on several occasions to clarifty the development.

VI.A.1. Reduced State Space, S_r

Since the superposition process events are associated only with certain transitions, it is desireable to focus attention on a subset of the state space. A new process may be defined which has as a state space those elements of S which correspond to at least one component process being in stage one.

Example VI.A.1 (continued). In the E(3,2) process, the reduced state space, denoted S_r, takes the form

$$S_r = \{(1,1),(1,2),(2,1),(3,1),(1,3)\} \ .$$

In S, (6.A.1.1), the path [(1,1):(1,2):(2,2):(3,2):(1,2)] in Figure VI.A.1 would correspond to the transition [(1,1):(1,2)] in the reduced space, S_r. Transition time would be governed by $\gamma_t(4,2\lambda)$. Transition probability is

$$\Pr[(1,1):(1,2)] = N/2^4 ,$$

where N=3 is the number of paths in S starting in state (1,1) and terminating with an event in state (1,2), having no intervening events.

Continuing with the general presentation, there are two advantages to the use of the reduced state space, S_r. First, the cardinality of S is $|S|=k^p$. Although $|S_r|$ grows rapidly with k and p, it is much smaller. An expression for $|S_r|$ is not given because it is rather complicated, and the next step in the state space reduction process renders it unnecessary. The main value of state space reduction is that if the transition pattern in the reduced state space has retained the Markov property, then by considering the transition times in S_r as events in a point process, we have a semi-Markov generated point process as defined in Chapter V.

Clearly, the Markov property is retained through the reduction process. That is, the transition probabilities associated with any state s in S_r are independent of the evolution of the state space prior to entering s.

VI.A.2. Transition Probabilities in S_r

Examine the direct transition in S_r from $\underline{u}^0=(s_1,\ldots,s_p)$ to $\underline{u}^1=(s'_1,\ldots,s'_{p-1},1)$. This transition corresponds to a path in S

$$\underline{v}^0 = (s_1,\ldots,s_p):\cdots:(s'_1,\ldots,s'_{p-1},k):(s'_1,\ldots,s'_{p-1},1) = \underline{v}^r \quad , \tag{6.A.2.1}$$

with all intervening states $\underline{v}^i$, i=1,...,r-1, satisfying the inequality

$$\underline{v}^i \geq \underline{v}^{i-1} \quad , \tag{6.A.2.2}$$

where it is understood that the inequality holds for each element of the vectors. The inequality condition (6.A.2.2) precludes an intervening event in the path (6.A.2.1).

Recall that in S, the choice of the state following any initial state is an event of probability 1/p. The number of transitions from $\underline{v}^0$ to $\underline{v}^{r-1}$ is

$$r-1 = \sum_{j=1}^{p-1} (s'_j-s_j) + (k-s_p) \tag{6.A.2.3}$$

(The process p was chosen arbitrarily as the source of the event. Any other process would have the same property, but would be more cumbersome notationally.) The number of paths in S satisfying (6.A.2.1) and (6.A.2.2) is given by the multinomial coefficient

$$N = \frac{(r-1)!}{\{\prod_{j=1}^{p-1} (s_j!-s_j)!\}(k-s_p)!} \qquad (6.A.2.4)$$

Combining (6.A.2.3) and (6.A.2.4), and noting that $Pr[\underline{v}^{r-1}:\underline{v}] = 1/p$ gives

$$Pr[\underline{u}^0:\underline{u}^1] = N/p^r \quad . \qquad (6.A.2.5)$$

Further, the transition time associated with this event is governed by

$$f_{\underline{u}^0:\underline{u}^1}(t) = \gamma_t(r,p\lambda) \qquad (6.A.2.6)$$

since $(\underline{u}^0:\underline{u}^1)$ represents r realizations of iid exponential $(p\lambda)$ random variables.

Example VI.A.1 (continued). In the reduced state space, S_r, associated with the E(k,p) process, the matrices of transition probabilities and transition time densities can be developed using (6.A.2.5) and (6.A.2.6). These matrices are shown in Figure VI.A.2 and Figure VI.A.3 respectively.

VI.A.3. The Collapsed State Space, S_c

Study of Figures VI.A.1, VI.A.2 and VI.A.3 reveals a degree of symmetry which suggests a further reduction in the size of the state space. In particular, it is possible to treat all permutations of a set of permissible stage

	(11)	(12)	(21)	(13)	(31)
(11)	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$
(12)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$
(21)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$
(13)	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
(31)	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{8}$

Figure VI.A.2 Transition matrix for the E(3,2) process; Reduced state space (S_r)

The states are denoted (ij). A matrix element is the probability the next "boxed" state (See Fig. VI.A.1) entered is (ij) given the previous "boxed" state is (kn).

	(11)	(12)	(21)	(13)	(31)
(11)	$\gamma_t(3,2\lambda)$	$\gamma_t(4,2\lambda)$	$\gamma_t(4,2\lambda)$	$\gamma_t(5,2\lambda)$	$\gamma_t(5,2\lambda)$
(12)	$\gamma_t(2,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(4,2\lambda)$	$\gamma_t(4,2\lambda)$
(21)	$\gamma_t(2,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(4,2\lambda)$	$\gamma_t(4,2\lambda)$
(13)	$\gamma_t(1,2\lambda)$	0	$\gamma_t(2,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(3,2\lambda)$
(31)	$\gamma_t(1,2\lambda)$	$\gamma_t(2,2\lambda)$	0	$\gamma_t(3,2\lambda)$	$\gamma_t(3,2\lambda)$

Figure VI.A.3 Transition time densities for the E(3,2) process; reduced state space (S_r)

The state representation is as in Fig. VI.A.2. Given that the system has just entered state (kn) and the next "boxed" state (Fig. VI.A.1) is (ij), the time of transition will be governed by the Erlang density indicated.

values as a single state denoted by a generic element. Thus $w=\{(s_1,\ldots,s_p)\}$ would represent all states of S_r which are distinct permutations of the elements of w. The collapsed state space is S_c, the partition of S_r defined by the sets w. The states of S_c will be denoted by lower case script letters indicating their roles as sets of states from S_r. States of S_r will still be referred to in vector notation as underlined lower case letters.

In Example VI.A.1, $S_c=\{\ 1,1\ ,\ 1,2\ ,\ 1,3\ \}$.

The cardinality of S_c is given by

$$|S_c| = \binom{k+p-2}{p-1} \quad , \tag{6.A.3.1}$$

the figurate number for k types of elements taken in sets of size p-1. Figure VI.A.4 compares $|S|$ with $|S_c|$ for k and p taking values 2 through 5.

To be useful in view of the results of Chapter V, transitions within the collapsed state space, S_c, must exhibit the Markov property. That is, it is necessary to show that S_r with its transition probability matrix, T_r, is a lumpable Markov chain with respect to the partition S_c.

Following Kemeny and Snell (1960, Ch. 4), a Markov chain, C, is said to be lumpable with respect to a partition $A=\{A_1,\ldots,A_r\}$ if for every starting vector $\underline{\pi}$, the lumped process is a Markov chain with transition probabilities independent of $\underline{\pi}$. A necessary and sufficient condition that C be lumpable with respect to the partition A is that for

k \ p	2	3	4	5
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125

Table of $|S| = k^p$

k \ p	2	3	4	5
2	2	3	4	5
3	3	6	10	15
4	4	10	20	35
5	5	15	35	70

Table of $|S_c| = \binom{k+p-2}{k-1}$

Figure VI.A.4 Comparison of state space size for the E(k,p) representations S and S_c. The dimension of the stage transition Markov chain is $|S| = k^p$. The dimension of the reduced, collapsed state space is $|S_c| = \binom{k+p-2}{k-1}$

every pair of sets A_i, A_j, $pr\{n:A_j\} = \sum_{m\in A_j} Pr\{n:m\}$ has the same value for every state n contained in A_i. These common values, $\{t_{ij}\}$, form the transition matrix for the lumped chain, A.

Formally, we state

Theorem VI.A.1: The Markov chain S_r is lumpable with respect to the partition S_c.

Proof: It is sufficient to show that

$$Pr[\underline{u}=(s_1,\ldots,s_p):w=\{s_1',\ldots,s_p'\}] = \text{constant}$$

for $\underline{u}\varepsilon\{(s_1,\ldots,s_p)\}$.

First, let u and w be generic names of elements of S_c and assume that a direct transition from u to w is feasible in S_r. That is, there exist elements $\underline{u}$ and $\underline{w}$ in u and w and a path in S, $(\underline{u}:\underline{v}_1:\ldots:\underline{v}_{r-1}:\underline{w})$ satisfying the inequality (6.A.2.2). Further, suppose process m is that process which records an event in the transition from $\underline{u}$ to $\underline{w}$. Then the number of steps from $\underline{u}$ to $\underline{w}$ is given by (6.A.2.3) as

$$r = \sum_{j\neq m} (s_j'-s_j) + (k+1-s_m)$$

$$= \sum_{j=1}^{p} s_j' - \sum_{j=1}^{p} s_j + 1 \quad , \qquad (6.A.3.2)$$

since $s_m'=k$ by construction. Thus, the number of transitions

in S corresponding to a transition from an element of u to an element of w is independent of the elements chosen. Hence,

$$\Pr[\underline{u}_i : w] = N_i / p^r, \quad i = 1,\ldots,n, \tag{6.A.3.3}$$

where N_i is the number of distinct paths in S satisfying (6.A.2.2) beginning at $\underline{u}_i$ and terminating at an element of w. Here, n represents the number of distinct permutations of the elements of u.

Let N_{ij} be the number of paths from $\underline{u}_i \in u$ to $\underline{w}_j \in w$, where again process m has recorded the transition event. Then

$$N_{ij} = \frac{(n-1)!}{\{\prod_{j \neq m} (s_j! - s_j)!\}(k - s_m)!}, \quad j = 1,\ldots,t; \; i = 1,\ldots,n \tag{6.A.3.4}$$

and

$$N_i = \sum_{j=1}^{t} N_{ij}, \tag{6.A.3.5}$$

where t is the number of distinct permutations of w.

Let $\underline{u}_i^* = (s_1^*,\ldots,s_p^*)$ be any other element of u. Then there exists a one to one mapping, h, of $\underline{u}_i$, onto $\underline{u}_i^*$ such that $h(s_j) = s_i^*$. Then a new element $\underline{w}_j^* = (s_1^*,\ldots,s_p^*)$ is formed from $h(s_j!) = s_i^*!$. It follows that $N_{ij}^* = N_{ij}$ and $N_i^* = N_i = N(u,w)$. Hence the lumpability condition is satisfied, thus completing the proof of Theorem VI.A.1.

The transition probability matrix for the collapsed state space is

$$T = \{Pr[u:w] | u,w \epsilon S_c\}$$

$$= \{ \sum_{\underline{w} \epsilon w} Pr[\underline{u}:\underline{w}] | u,w \epsilon S_c \} \quad . \qquad (6.A.3.6)$$

The transition time density matrix is

$$F(t) = \{\gamma_t(N(u,w),p\lambda) | u,w \epsilon S_c\} \qquad (6.A.3.7)$$

Returning once more to Example VI.A.1, Figures VI.A.5 and VI.A.6 show the matrices of transition probabilities and transition densities, extracted from Figures VI.A.2 and VI.A.3. For example,

$$Pr[\{1,2\},\{1,2\}] = Pr[(1,2): (1,2)] + Pr[(1,2): (2,1)]$$

$$= 3/8$$

VI.A.3.a. Proof of Theorem III.A.2

Theorem III.A.2 claimed the interval spectral density of an E(k,p) process is expressible as the ratio of finite, rational polynomials in cos ω. Theorem V.B.3 showed the same result for the interval spectral density of a univariate semi-Markov generated point process. In this section it has been shown that an E(k,p) process has an

	{11}	{12}	{13}
{11}	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$
{12}	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$
{13}	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

Figure VI.A.5 Transition matrix for the E(3,2) process; collapsed state space (S_c).

The states are denoted {ij} to indicate their functions as sets of states in S_p. That is {ij} = {(ij),(ji)}. As in Fig. VI.A.2, transitions represent movement between "boxed" states in Fig. VI.A.1.

	{11}	{12}	{21}
{11}	$\gamma_t(3,2\lambda)$	$\gamma_t(4,2\lambda)$	$\gamma_t(5,2\lambda)$
{12}	$\gamma_t(2,2\lambda)$	$\gamma_t(3,2\lambda)$	$\gamma_t(4,2\lambda)$
{21}	$\gamma_t(1,2\lambda)$	$\gamma_t(2,2\lambda)$	$\gamma_t(3,2\lambda)$

Figure VI.A.6 Transition time densities for the E(3,2) process; collapsed state space (S_c).

State space representation as in Fig. VI.A.5.
Interpretations of entries as in Fig. VI.A.3.

equivalent representation as a semi-Markov generated point process, thus establishing Theorem III.A.2 as a special case of Theorem V.B.3.

VI.A.4. The E(2,2) Process as a Semi-Markov Generated Point Process

Let $N_i(t)$, $i = 1,2$ be a point process with independent identically distributed intervals governed by the $\gamma_t(2,\lambda)$ density. Let $N(t) = N_1(t) + N_2(t)$ denote an E(2,2) counting process. $N_i(t)$ may be thought of as a Poisson process in which only alternate events are recorded. Consider the process $N_i(t)$ to be in stage one if its next event will not be recorded, and stage two otherwise. Events in the superposed process, N(t), will occur whenever either contributing process makes a transition from stage two to stage one.

Define the states of the semi-Markov process by

$$(i,j) = \{N_1(t) \text{ is in stage } i;\ N_2(t) \text{ is in stage } j\},$$
$$i,j = 1,2,$$

so that the entire state space is

$$S = \{(1,1),(1,2),(2,1),(2,2)\}.$$

Assume that at some time the process is in state (i,j). Let V_1 and V_2 be the times to the next transition in processes 1 and 2 respectively. Then, by hypothesis,

V_1 and V_2 are iid obeying the exponential probability law with rate λ. Component process 1 will advance before component 2 if $V_1 < V_2$, an event of probability 1/2. The time until the next transition will be $V = \min\{V_1, V_2\}$ which is distributed exponentially at rate 2λ.

Transitions within the chain S follow the pattern illustrated in Fig. VI.A.7. From any state, one of two paths will be chosen with equal probability. Transitions to the left result in events of N(t). The possible trajectories from each state terminating in a main process event are enumerated in Fig. VI.A.8.

The reduced state space of the E(2,2) process is

$$S_r = \{(1,1),(1,2),(2,1)\}.$$

The transition matrix for the reduced state space may be derived from Fig. VI.A.8. For example,

$$\Pr[(1,1): (1,1)] = \Pr[\text{trajectory 1}] + \Pr[\text{trajectory 2}]$$

$$= 1/2.$$

Similarly, the matrix of transition time densities may be taken from the table of Fig. VI.A.8. Figure VI.A.9 lists the transition probability and interval density matrices.

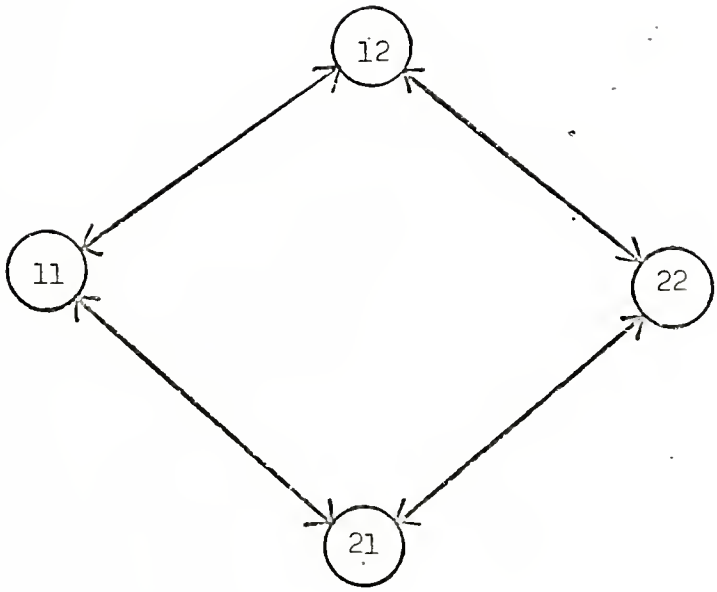


Figure VI.A.7 Transition network for the E(2,2) process. Transitions to the left result in events recorded in the point process. Transitions to the right do not have corresponding events.

No.	Trajectory	Probability	Distribution
1	(11):(12):(11)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
2	(11):(21):(11)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
3	(11):(12):(22):(12)	$\frac{1}{8}$	$\gamma_t(3,2\lambda)$
4	(11):(12):(22):(21)	$\frac{1}{8}$	$\gamma_t(3,2\lambda)$
5	(11):(21):(22):(12)	$\frac{1}{8}$	$\gamma_t(3,2\lambda)$
6	(11):(21):(22):(21)	$\frac{1}{8}$	$\gamma_t(3,2\lambda)$
7	(12):(11)	$\frac{1}{2}$	$\gamma_t(1,2\lambda)$
8	(21):(11)	$\frac{1}{2}$	$\gamma_t(1,2\lambda)$
9	(12):(22):(12)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
10	(12):(22):(21)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
11	(21):(22):(12)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
12	(21):(22):(21)	$\frac{1}{4}$	$\gamma_t(2,2\lambda)$
13	(22):(12)	$\frac{1}{2}$	$\gamma_t(1,2\lambda)$
14	(22):(21)	$\frac{1}{2}$	$\gamma_t(1,2\lambda)$

Figure VI.A.8 Stage transition trajectories in the E(2,2) process. This is an enumeration of all trajectories from any state in the stage transition system, S, terminating with an event in the point process (a transition to the left in Fig. VI.A.7). The probability and distribution columns are conditioned on knowledge of the initial state.

	(11)	(12)	(21)
(11)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
(12)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
(21)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Transition Matrix

(11)

	(11)	(12)	(21)
(11)	$\gamma_t(2, 2\lambda)$	$\gamma_t(3, 2\lambda)$	$\gamma_t(3, 2\lambda)$
(12)	$\gamma_t(1, 2\lambda)$	$\gamma_t(2, 2\lambda)$	$\gamma_t(2, 2\lambda)$
(21)	$\gamma_t(1, 2\lambda)$	$\gamma_t(2, 2\lambda)$	$\gamma_t(2, 2\lambda)$

Transition Time Densities

Figure VI.A.9 Reduced state space (S_r) matrices for the E(2,2) process.

Interpretations of these matrices are as in Figures VI.A.2 and VI.A.3.

Finally, lumping states (2,1) and (1,2) forms the collapsed state space

$$S_c = \{\{1,1\},\{1,2\}\} \qquad (6.A.4.1)$$

with transition matrix

$$T = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \qquad (6.A.4.2)$$

and interval density matrix

$$F(t) = \begin{pmatrix} \gamma_t(2,2\lambda) & \gamma_t(3,2\lambda) \\ \gamma_t(1,2\lambda) & \gamma_t(2,2\lambda) \end{pmatrix} . \qquad (6.A.4.3)$$

Clearly, the stationary vector of T is

$$\underline{\pi}^T = (1/2,\ 1/2) . \qquad (6.A.4.4)$$

From Lemma V.B.1, the density of an arbitrary interval is

$$f_o(t) = \underline{\pi}^T F(t) \underline{1}$$

$$= \{\gamma_t(1,2) + 2\gamma_t(2,2) + \gamma_t(3,2)\}/4. \qquad (6.A.4.5)$$

In Chapter VII, this result is shown to be the same as that derived from the expression for the density of an arbitrary superposition interval.

Lewis, et al (1973) showed that for the E(2,2) process, the serial interval covariance vanished for lags greater than one. The semi-Markov representation admits the following

Theorem VI.A.2: Let X_n be the n-th interval in an E(2,2) process, $n = 0,\pm1,\pm2,\ldots$. Then X_n is independent of all realizations X_j, $j < n-1$. That is to say

$$\Pr\{X_n \leq t | X_{n-2} = t_2, X_{n-3} = t_3, \ldots\} = \Pr\{X_n \leq t\}$$

Proof: The interval X_{n-2} was terminated by the transition of one of the component processes from stage 2 to stage 1 putting the system in state (1,1), (1,2) or (2,1). Similarly, the interval X_n will begin with the system in state (1,1), (1,2) or (2,1), i.e., states {1,1} or {1,2} of S_c. Assuming X_{n-2} terminated by a transition to (1,1), (6.A.4.2) shows that states {1,1} or {1,2} are equally likely for the commencement of interval X_n. This is also the case for X_{n-2} terminating by entry into states (1,2) or (2,1). Thus information regarding the state of the system derived from the interval X_{n-2} provides no information regarding the state of the system at the outset of X_n, hence X_n and X_{n-2}

are independent. Similarly, all knowledge of the system derived from knowledge of the events X_{n-2}, X_{n-3},..., sheds no light on the distribution of X_n.

This completes the proof of Theorem VI.A.2.

To show that a first lag dependence exists, it is sufficient to establish that $\text{cov}(X_o, X_1) \neq 0$. To use Theorem V.B.1 which formulates the serial covariance of a semi-Markov generated point process, it is necessary to determine the matrix of weighted first moments, M_1. Let U be a random variable with density $\gamma_t(k,\lambda)$. Then $E[U] = k/\lambda$ and $E[U^2] = k(k+1)/\lambda^2$.

Thus, using (6.A.4.2) and (6.A.4.3)

$$M_1 = \begin{pmatrix} 1/2 & 3/4 \\ 1/4 & 1/2 \end{pmatrix}$$

Conventionally, $T^0 = I$, and in this process

$$T^\infty = T,$$

so

$$\text{cov}(X_o, X_1) = \underline{\pi}^T M_1 (I - T^\infty) M_1 \underline{1}$$

$$= -1/16\ \lambda^2$$

The first serial correlation of this process is given by

$$\rho_1 = \mathrm{cov}(X_o, X_1)/\mathrm{Var}(X_o) \quad .$$

Lemma V.B.2 gives the variance of the process. The matrix of second moments is (from (6.A.4.2) and (6.A.4.3))

$$M_2 = \begin{pmatrix} 3/4\ \lambda^2 & 3/2\ \lambda^2 \\ 1/4\ \lambda^2 & 3/4\ \lambda^2 \end{pmatrix} ,$$

so

$$\mathrm{Var}(X_o) = \underline{\pi}^T (M_2 - M_1 T^{\infty} M_1) \underline{1}$$

$$= 5/8\ \lambda^2 ,$$

and

$$\rho_1 = -1/10 ,$$

which is the result of Lawrence (1973) as well as Lewis, et al (1973). Note that while Lawrence gave the result that

$\rho_1 = -1/10$ for this process, we have established the stronger result of Theorem VI.A.1.

Theorem V.B.2 gives the spectral representation of the process. Computing first

$$(Ix-T)^{-1} = \begin{pmatrix} x-1/2 & 1/2 \\ 1/2 & x-1/2 \end{pmatrix} \Big/ x(x-1)$$

then, taking $i = \sqrt{-1}$,

$$f_+^{(2)}(\omega) = \{26+(15e^{i\omega}+1)e^{-i\omega}/(e^{i\omega}-1)+(15e^{-i\omega}+1)e^{i\omega}/(e^{-i\omega}-1)$$

$$= (5-\cos\ \omega)/5$$

which is the form presented by Lewis, et al (1973). There it was derived by the method outlined in Chapter III, Section A (above).

VI.A.5. Determination of T, and M_s for the Collapsed E(k,p) Semi-Markov Process

The ground work for determining the transition matrix, T, of the collapsed state space, S_c, was established in Subsections VI.A.2 and VI.A.3. Equations (6.A.2.3), (6.A.2.4) and (6.A.2.5) gave the one stage transition in the reduced state space, S_r, to be a modified multinomial random variable for states satisfying the relation (6.A.2.2).

The lumpability condition of Subsection VI.A.3 states that the sum of the direct transition probabilities in S_r from an element of a state set $\{(j_1,\ldots,j_p)\} \in S_c$ to each of the elements in $\{(j'_1,\ldots,j'_p)\}$ gives the collapsed space direct transition probability. In general, then, the method for determining $\Pr[\{(j_1,\ldots,j_p)\}:\{j'_1,\ldots,j'_p)\}]$ is to enumerate the elements of $\{(j'_1,\ldots,j'_p)\} = \{\underline{v}_1,\ldots,\underline{v}_s\}$, select a particular element $\underline{u} \in \{(j_1,\ldots,j_p)\}$ and evaluate (6.A.2.5) for all pairs $(\underline{u}:\underline{v}_i)$, summing the results.

The process outlined is cumbersome, although easier to implement than many of the computations associated with superposition processes. While there may be as many as p elements in a collapsed state set, this will only occur when the indices contained in a typical element of the set are all distinct. Generally, several component processes will be in the same stage, resulting in a multiplicity of indices in the collapsed state set.

Example VI.A.2: *Transition Probabilities in the E(5,2) Process.* In the collapsed state space of the E(5,2) process, what is $\Pr[\{1,1,2,2,2\}:\{1,1,1,2,3\}]$? Although there are 20 elements of $\{1,1,1,2,3\}$, only six satisfy the condition (6.A.2.2) with respect to the generic element (1,1,2,2,2). The condition is violated in all cases in which more than one component process must enter stage 1. For example

$$\Pr[(1,1,2,2,2):(1,2,3,1,1)] = 0.$$

Further, each of the six attainable states will be chosen with equal probability, since for example

$$\Pr[(1,1,2,2,2):(1,1,1,2,3)] = \Pr[(1,1,2,2,2):(1,1,3,2,1)]$$

Thus, using (6.A.2.3), (6.A.2.4) and (6.A.2.5),

$$\Pr[\{1,1,2,2,2\}:\{1,1,1,2,3\}] = 6(2)(5^{-3}) = 12/125$$

Continuing the general development, we examine the moments of the process. Determination of the M_s matrices is a byproduct of the T computation. In Chapter V, the matrix $E_s = \{\mu_{ij}^{(s)}\}$ was defined as the matrix of s-th moments of the random variables, S_{ij}, of a semi-Markov process. In a semi-Markov process corresponding to an E(k,p) process, the distribution of S_{ij} is totally determined by the rate λ, and the number of state transitions, r, from state i to state j given by (6.A.2.3), where i and j are indices assigned two of the state sets of S_c.

It is convenient at this juncture to establish

Lemma VI.A.1: Let X be a non-negative random variable with probability density function $\gamma_t(r,\lambda)$. Then

$$E[X^{k+1}] = \lambda^{-k-1}(r+k)!/(r-1)!$$

$$= \lambda^{-1}(r+k)E[X^k],\ k = 0,1,2,\ldots \qquad (6.A.5.1)$$

Proof: Formally operating on the definition of expectation of an Erlang distributed random variable

$$E[X^{k+1}] = \int_0^\infty (x^{k-1})\ \lambda^r\ \frac{x^{r-1}}{(r-1)!}\ e^{-\lambda x}\ dx$$

$$= \frac{(r+k)!}{(r-1)!}\ \lambda^{-k-1} \int_0^\infty \frac{\lambda^{r+k-1}\ x^{r+k}\ e^{-\lambda x}\ dx}{(r+k)!}$$

$$= \frac{(r+k)!}{(r-1)!}\ \lambda^{-k-1}$$

$$= \lambda^{-1}\ (r+k)E[X^k].$$

Taking $E[X^0] = 1$, so that $E[X] = r/\lambda$, completes the proof.

Returning to the computation of M_s, assign an index $j = 1,\ldots,W$ to each state set of S_c. Denote by r_{ij} the number of stages separating the states of i and j, where r_{ij} is given by (6.A.2.3). Then

$$E_{s+1} = \{(r_{ij}+s)!/(r_{ij}-1)!\}/\lambda^{s+1}$$

$$= \{(r_{ij}+s)\mu_{ij}^{(s)}\}/\lambda \qquad (6.A.5.2)$$

and

$$M_{s+1} = \{t_{ij}(r_{ij}+s)\mu_{ij}^{(s)}\}/\lambda \ . \qquad (6.A.5.3)$$

In particular,

$$M_1 = \{t_{ij}r_{ij}\}/\lambda \qquad (6.A.5.4)$$

and

$$M_2 = \{t_{ij}r_{ij}(r_{ij}+1)\}/\lambda^2 . \qquad (6.A.5.5)$$

The results of the computation proposed in this section and applied to selected E(k,p) processes are summarized in Appendix C.

VI.B. A SEMI-MARKOV GENERATED POINT PROCESS MODEL OF THE H(k,p;$\underline{q}$,$\underline{\lambda}$) PROCESS

Following the procedure of Section A, let $Y = X_j$ with probability q_j, $j = 1,\ldots,k$, where X_j is governed by the survivor function $R_j(x) = \exp\{-\lambda_j x\}$. Then Y has the hyperexponential survivor distribution $R(y) = \sum_{j=1}^{k} q_j \exp\{-\lambda_j x\}$.

Let $N_1(t)$ represent a counting process in which the times between events are iid random variables governed by R(y). Then $N_1(t)$ is said to be in stage j if the time to the next event in the process has the survivor distribution $R_j(y)$.

Let $N(t) = N_1(t) + \ldots + N_p(t)$. The state of the process $N(t)$ is the vector $(\nu_1,\ldots,\nu_p)$ where ν_j taking on values $1,\ldots,k$ corresponds to the stage of component process j. An event occurs in the process $N(t)$ whenever an event occurs in a component process. The state of $N(t)$ will change only in that element corresponding to the component process responsible for the event, and then only if the component process enters a new stage.

The time between events in the superposed process $N(t)$ will be the minimum of $S_i(\nu_i)$, $i = 1,\ldots,p$, where $S_i(\nu_i)$ is the time to the next event in process i given that it is currently in stage ν_i. Because of the lack of memory property of the exponential distribution, there is no need for concern with regard to forward recurrence time in this formulation.

The probability that component process m is responsible for the next event in $N(t)$ is

$$\Pr[S_m(\nu_m) = \min_i\{S_i(\nu_i)\}]$$

$$= \lambda(\nu_m)/\sum_{i=1}^{p} \lambda(\nu_i) \ , \qquad (6.B.1)$$

which is independent of all history except the current state of the system. Here $\lambda(\nu_j)$ is the parameter of the exponential distribution governing a process j in stage ν_j.

Thus, a semi-Markov process has been constructed which represents the superposition of p iid hyperexponential renewal processes. Transition from state $\underline{\nu} = (\nu_1,\ldots,\nu_p)$ to $\underline{\nu}' = (\nu_1',\ldots,\nu_p')$ is possible only if $\underline{\nu}$ and $\underline{\nu}'$ differ in at most one element. Let m be the index corresponding to the element of $\underline{\nu}'$ which differs from ν_i. Then, if $\nu_m = j$ and $\nu_m' = j'$,

$$\Pr[\nu:\nu'] = \frac{\lambda_j q_{j'}}{\sum_{i=1}^{p} \lambda(\nu_i)} \quad , \; j \neq j' \tag{6.B.2}$$

That is, the probability of transition from $\underline{\nu}$ to $\underline{\nu}'$ is the probability that the next event is from process m, and that process m enters stage j' following the event.

The probability that the system returns to its original state is given by

$$\Pr[\nu:\nu] = \frac{\sum_{i=1}^{p} \lambda(\nu_i) q(\nu_i)}{\sum_{i=1}^{p} \lambda(\nu_i)} \tag{6.B.3}$$

Example VI.B.1: *The H(2,3;$\underline{q}$,$\underline{\lambda}$) Process.* With k = 2 and p = 3, the survivor function is

$$R(t) = q_1 e^{-\lambda_1 t} + q_2 e^{-\lambda_1 t} \quad ,$$

and the state space is

$$S = \{(1,1,1),(1,1,2),(1,2,1),(2,1,1),(1,2,2),(2,1,2),(2,2,1),(2,2,2)\}$$

Figure VI.B.1 shows the network of possible transitions. Note that all arrows are double ended, and that each transition represents an event in the main process.

The sojourn times in each state are the minimums of exponential random variables, hence are themselves exponential. The mean time in state (i,j,ℓ) is $(\lambda_i + \lambda_j + \lambda_\ell)^{-1}$ for $i,j,\ell = 1,2$, where it is understood that the sojourn time is considered to be the time between events regardless of the state entered following an event. Transition probabilities, shown in Fig. VI.B.2 are computed from (6.B.3).

VI.B.1. The Collapsed State Space, S_c

Examination of Figures VI.B.1 and VI.B.2 suggest a collapsed state space might be appropriate to this problem. Let $u = \{(u_1,\ldots,u_k)\}$ be an element in the collapsed state space, S_c. Here u will represent all vectors $\underline{v} \in S$ for which u_j elements of v have the value j, $j = 1,\ldots,k$. In Example VI.B.1, the following correspondence applies:

$$u = \{3,0\} < > \underline{v} = (1,1,1)$$

$$u = \{1,2\} < > \underline{v} = (1,2,2),(2,1,2),(2,2,1)$$

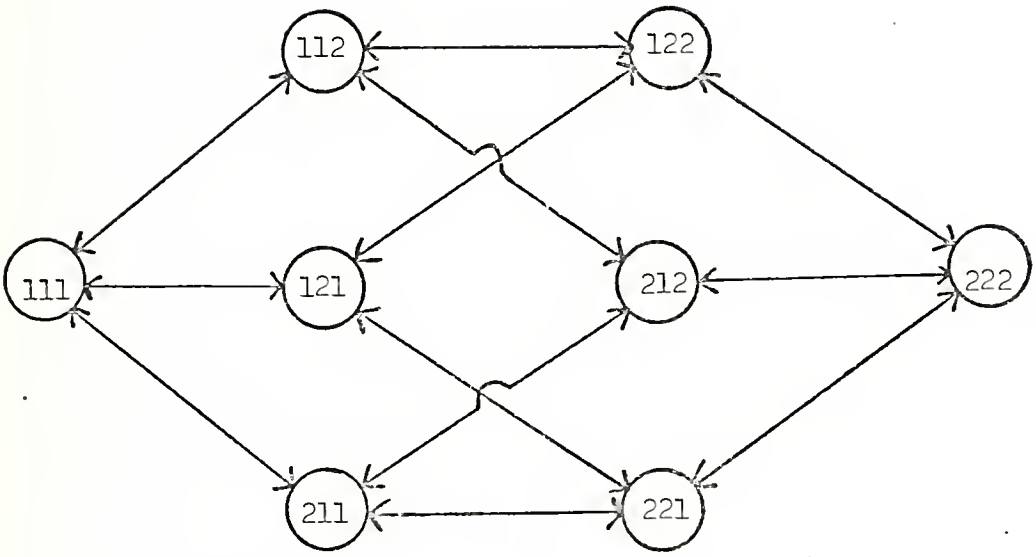


Figure VI.B.1 Transition network for the stage transitions in an H(2,3;$\underline{q}$,$\underline{\lambda}$) process.

In this system, all stage transitions are recorded as events in the point process. In addition, it is possible to have direct transitions from a state to itself, resulting in a point process event.

	(111)	(112)	(121)	(211)	(122)	(212)	(221)	(222)
(111)	q_1	$\frac{q_2}{3}$	$\frac{q_2}{3}$	$\frac{q_2}{3}$	0	0	0	0
(112)	$\frac{\lambda_2 q_1}{\lambda_2+2\lambda_1}$	$\frac{2\lambda_1 q_1+\lambda_1 q_2}{\lambda_2+2\lambda_1}$	0	0	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	0	0
(121)	$\frac{\lambda_2 q_1}{\lambda_2+2\lambda_1}$	0	$\frac{2\lambda_1 q_1+\lambda_2 q_2}{\lambda_2+2\lambda_1}$	0	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	0	0
(211)	$\frac{\lambda_2 q_1}{\lambda_2+2\lambda_1}$	0	0	$\frac{2\lambda_1 q_1+\lambda_2 q_2}{\lambda_2+2\lambda_1}$	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	$\frac{\lambda_1 q_2}{\lambda_2+2\lambda_1}$	0	0
(122)	0	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	0	$\frac{2\lambda_2 q_2+\lambda_1 q_1}{\lambda_1+2\lambda_2}$	0	0	$\frac{\lambda_1 q_2}{\lambda_1+2\lambda_2}$
(212)	0	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	0	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	0	$\frac{2\lambda_2 q_2+\lambda_1 q_1}{\lambda_1+2\lambda_2}$	0	$\frac{\lambda_1 q_2}{\lambda_1+2\lambda_2}$
(221)	0	0	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	$\frac{\lambda_2 q_1}{\lambda_1+2\lambda_2}$	0	0	$\frac{2\lambda_2 q_2+\lambda_1 q_1}{\lambda_1+2\lambda_2}$	$\frac{\lambda_1 q_2}{\lambda_1+2\lambda_2}$
(222)	0	0	0	0	$\frac{q_1}{3}$	$\frac{q_1}{3}$	$\frac{q_1}{3}$	q_2

Figure VI.B.2 Transition matrix for the stage transition system of the $H(2,3;\underline{q},\underline{\lambda})$ process, (S).

The states are designated (ijk), with positive matrix entries corresponding to arcs in Fig. VI.B.1. plus direct return.

$$u = \{2,1\} \longleftrightarrow \underline{v} = (1,1,2),(1,2,1),(2,1,1)$$

$$u = \{0,3\} \longleftrightarrow \underline{v} = (2,2,2)$$

Formally,

$$S_c = \{(u_1,\ldots,u_k) | u_j \varepsilon \{0,\ldots,p\}, \Sigma u_j = p\}. \qquad (6.B.1.1)$$

The cardinality of S_c is the same as the number of distinct sets of k objects taken in groups of size p, i.e., the figurate number

$$|S_c| = N = \binom{k+p-1}{p} .$$

This compares favorably with the cardinality of S, $|S| = k^p$. For example, if $k = 5$, and $p = 3$ then $|S| = 125$ while $|S_c| = 35$.

Two vectors u,u' in S_c are said to be <u>adjacent</u> if $\sum_{j=1}^{k} |u_j - u'_j| \leq 2$ and $\sum_{j=1}^{k} (u_j - u'_j) = 0$. Adjacent states in S_c have corresponding states in S which differ in at most one element, hence direct transitions in S_c are possible only between adjacent states.

To establish that the Markov property may be ascribed to S_c, recall from Section VI.A.3 that it is sufficient that S_c partitions S in such a way that for each $\underline{v}$ in S represented by u in S_c,

$$\Pr[u:u'] = \sum_{\underline{v}'\in v'} \Pr[\underline{v}:v'] = \text{constant}$$

For definiteness, assume that a transition from u to u' results from a component process in stage i transiting to stage j. In u, there are u_i processes in stage i, and the probability that one of these processes generates the next event is

$$u_i\lambda_i / \sum_{n=1}^{k} u_n\lambda_n$$

which corresponds to summing the expression (6.B.1.1) for the probability that a particular process is responsible for the next event over all processes in stage i. The probability that the chosen process will enter stage j is q_j, leading to

$$\Pr[u:u'] = u_i\lambda_i q_j / \sum_{n=1}^{k} u_n\lambda_n \qquad (6.B.1.2)$$

irrespective of the state of S chosen to represent u.

Continuing the Example VI.B.1, the transition matrix of the collapsed $H(2,3;\underline{q},\underline{\lambda})$ process is shown in Fig. VI.B.3.

VI.E.2. Expectation Matrices for the $H(k,p,\underline{q},\underline{\lambda})$ Semi-Markov Process

As observed above, the sojourn time in a state $u \in S_c$ is exponential with parameter $\theta_u = \sum_{i=1}^{k} u_i\lambda_i$ independent

	30	21	12	03
30	q_1	q_2	0	0
21	$\frac{\lambda_2 q_1}{\lambda_2+2\lambda_1}$	$\frac{2\lambda_1 q_1+\lambda_2 q_2}{\lambda_2+2\lambda_1}$	$\frac{2\lambda_1 q_2}{\lambda_2+2\lambda_1}$	0
12	0	$\frac{2\lambda_2 q_1}{\lambda_1+2\ 2}$	$\frac{2\lambda_1 q_1+\lambda_2 q_2}{\lambda_2+2\lambda_1}$	$\frac{\lambda_1 q_2}{\lambda_1+2\lambda_2}$
03	0	0	q_1	q_2

Figure VI.B.3 Transition Matrix for the H(2,3;$\underline{q}$,$\underline{\lambda}$) process; collapsed state space, S_c.

The states in this process are denoted {ij} to indicate their role as partitioning sets of the state space, S. For example, {21} = {(112),(121),(211)}; {12} = {(122),(212),(221)}.

of the subsequent state. By Lemma VI.A.1, if X has probability density function $f(t) = \lambda e^{-\lambda t} = \gamma_t(1,\lambda)$, then

$$E[X^r] = r!/\lambda^r, \quad k = 0,1,2,... \qquad (6.B.1.3)$$

Indexing the sets of S_c on the positive integers so that θ_i is the parameter of the exponential sojourn time distribution in state u_i, let E_r be the matrix of expectations $\mu_{ij}^{(r)}$ of sojourn time in state i conditioned upon a direct transition to state j. Then

$$\mu_{ij}^{(r)} = r!/\theta_i^r, \text{ if i adjacent j,}$$

and is not defined but may be taken to be 0 otherwise.

The matrix M_r of weighted moments is

$$M_r = \{\mu_{ij}^{(r)} \; t_{ij}\}$$

where the transition matrix of the collapsed process is

$$T = \{t_{ij}\}.$$

This method is used in Section VIII.B.1 to generate numerical spectral representations for the $E(2,2;\underline{q},\underline{\lambda})$ process.

VII. THE DENSITY OF AN ARBITRARY INTERVAL IN THE SUPERPOSITION OF RENEWAL PROCESSES

The probability density function of an arbitrary interval in a superposition point process can provide some initial information regarding the component processes. This chapter examines the pdf of an arbitrary superposition interval as a function of the component interval probability density function. Although the material presented is not new, it is given here to provide an added characterization of the Erlang and hyperexponential processes.

In Section A, a general component process interval pdf is assumed, and the form of the marginal superposition density is determined. It is shown that the initial value of the density is a function of the number of component processes.

Sections B and C apply the results of Section A to the $E(k,p)$ and $H(k,p;\underline{q},\underline{\lambda})$ processes, respectively.

VII.A. THE ARBITRARY INTERVAL DENSITY IN THE SUPERPOSITION OF RENEWAL PROCESSES

The purpose of this section is to derive heuristically the probability density function of an arbitrary interval in the superposition of p iid renewal processes. This result was originally given by Cox and Smith (1954). Let $R(t) = Pr\{X>t\}$ be the survivor function of the intervals of of the component renewal processes, with associated pdf $f(t)$.

Then the equilibrium excess distribution (forward recurrence time distribution) is (Cox and Lewis, 1966)

$$Z(t) = \lim_{u \to -\infty} \Pr\{\text{time to next event} > t \mid \text{event at } u\}$$

$$= (E[X])^{-1} \int_t^{\infty} R(v)dv \ .$$

The length of an arbitrary interval in the superposition process is the minimum of the time to the next event in each of the p component processes. That is, if $X^{(p)}$ denotes an arbitrary interval of the superposition process,

$$R^{(p)}(t) = \Pr[X^{(p)} > t] = \prod_{j=1}^{p} \Pr[X_j > t] \ ,$$

where X_j is the time to the next event in the j-th component process.

Let s be the index of the component responsible for the most recent event. Then

$$\Pr[X_j > t] = \begin{cases} R(t), & j = s \ , \\ Z(t), & j \neq s \ , \end{cases}$$

so,

$$R^{(p)}(t) = R(t)Z^{p-1}(t) \qquad (7.A.1)$$

The intervals of the component processes have pdf

$$f(t) = -\frac{dR}{dt}(t) \quad ,$$

so the pdf of an arbitrary superposition interval is

$$f^{(p)}(t) = -\frac{dR^{(p)}}{dt}(t)$$

$$= \{f(t)Z(t) + (p-1)\frac{R^2(t)}{E[X]}\}Z^{p-2}(t) \quad . \qquad (7.A.2)$$

The initial point of the arbitrary interval density is closely related to the number of component processes, as can be seen from the following

Theorem VII.A.1: Let $f(t)$ and $f^{(p)}(t)$ be as defined in this section. Then

$$f^{(p)}(0+) = f(0+) + (p-1)/E[X] \quad , \qquad (7.A.3)$$

and in particular, when $f(0+) = 0$ and $E[X^{(p)}] = 1$, then

$$f^{(p)}(0+) = (p-1)/p \quad . \qquad (7.A.4)$$

Proof: From the definition of $Z(t)$ as the survivor function of a non-negative random variable it is clear that $Z(0) = 1$. Since $R(t)$ is also the survivor function of a non-negative random variable, $R(0) = 1$. Thus, in (7.A.2),

$$\lim_{t\to 0} f^{(p)}(t) = f(0+) + (p-1)/E[X] \ .$$

For a stationary process, $E[X^{(p)}] = E[X]/p$, so if $E[X^{(p)}] = 1$, then $E[X] = p$. Hence for $f(0+) = 0$, and $E[X^{(p)}] = 1$,

$$f^{(p)}(0+) = (p-1)/p \ .$$

This completes the proof of Theorem VII.A.1.

The final form of this result is related to a similar result of Cox and Lewis (1966, Ch. 8). Recalling from (2.B.2.5) that

$$m_f^{(p)}(t) = m_f(t) + (p-1)m \ ,$$

and observing that as t grows large the dependence on the arbitrary event diminishes and $m_f(t)$ tends toward m, so that

$$m_f^{(p)}(\infty) = pm \ .$$

Thus if the component processes have $m_f(0+) = 0$, then

$$m_f^{(p)}(0+)/m_f^{(p)}(\infty) = (p-1)/p \ . \qquad (7.A.5)$$

VII.B. THE ARBITRARY INTERVAL pdf FOR THE E(k,p) PROCESS

In the E(k,p) process, the component interval pdf is

$$\gamma_t(k,\lambda) = \lambda^k t^{k-1} e^{-\lambda t}/(k-1)! \quad . \tag{7.B.1}$$

The survivor function is

$$R_k(t) = \int_t^\infty \gamma_u(k,\lambda)\, du$$

$$= \lambda^{-1} \sum_{j=1}^{k} \gamma_t(j,\lambda) \quad , \tag{7.B.2}$$

and the equilibrium excess distribution is

$$Z_k(t) = (\lambda/k) \int_t^\infty R_k(u)\, du$$

$$= (\lambda k)^{-1} \sum_{j=1}^{k} (k+1-j)\, \gamma_t(j,\lambda) \quad . \tag{7.B.3}$$

Substituting (7.B.1), (7.B.2) and (7.B.3) in (7.A.2) yields

$$f_k^{(p)}(t) = \{\gamma_t(k,\lambda) \sum_{j=1}^{k} (k+1-j)\, \gamma_t(j,\lambda)$$

$$+ (p-1)/(k\lambda)[\sum_{j=1}^{k} \gamma_t(j,\lambda)]^2\}$$

$$\times \{\sum_{j=1}^{k} (k+1-j)\, \gamma_t(j,\lambda)\}^{p-2} \quad . \tag{7.B.4}$$

Computation of the pdf may be simplified by implementing the identity

$$\gamma_t(j,n\lambda)\ \gamma_t(k,m\lambda)$$

$$= \frac{\lambda n^j m^k}{(n+m)^{j+k-1}} \binom{j+k-2}{j-1} \gamma_t(j+k-1, m\lambda+n\lambda) \quad . \qquad (7.B.5)$$

Thus the product of Erlang probability density functions is a scaled Erlang pdf, so $f_k^{(p)}(t)$ is a linear convex combination of Erlang probability density functions.

Using (7.B.5), (7.B.4) can be expressed as

$$f_k^{(p)}(t) = (\lambda k)^{1-p} \{ \sum_{j=1}^{k} (k+1-j)\ \gamma_t(j,\lambda) \}^{p-2}$$

$$\times\ 2\lambda \{ \sum_{j=1}^{k} 2^{-(k+j)} (k+1-j) \binom{k+j-2}{k-1} \gamma_t(j+k-1, 2\lambda)$$

$$+ (p-1) \sum_{j=1}^{k} 4^{-j} \binom{2j-2}{j-1} \gamma_t(2j-1, 2\lambda)$$

$$+ 2(p-1) \sum_{j=1}^{k-1} \sum_{n=j+1}^{k} 2^{-(j+n)} \binom{j+k-2}{j-1} \gamma_t(j+n-1, 2\lambda) \} \ ,$$

(7.B.6)

$$= \sum_{j=1}^{N} \alpha_j \gamma_t(j, p\lambda) \quad , \qquad (7.B.7)$$

where $N = p(k-1)+1$, $\sum_{j=1}^{N} \alpha_j = 1$ and $\alpha_j \geq 0$, $j=1,\ldots,N$.

Applying Theorem VII.A.1, and observing from (7.B.1) that for $k>1$, $\gamma_{0+}(k,\lambda) = 0$, and with $E[X] = k/\lambda$, then

$$f_k^{(p)}(0+) = \lambda(p-1)/k$$

$$= (p-1)/p \ , \quad \text{for } \lambda = k/p \ . \qquad (7.B.8)$$

Applying the formula (7.B.6) to the E(2,2) process yields

$$f_2^{(2)}(t) = \{(1/2)\gamma_t(2,2\lambda) + (1/4)\gamma_t(3,2\lambda)$$
$$+ (1/2)\gamma_t(1,2\lambda) + (1/4)\gamma_t(3,2\lambda)$$
$$+ (1/2)\gamma_t(2,2\lambda)\}/2$$

$$= (1/4)\{\gamma_t(1,2\lambda) + 2\gamma_t(2,2) + \gamma_t(3,2\lambda)\}$$

which is identical with the probability density function derived for this process using the semi-Markov generated point process representation (6.A.4.5).

Using (7.B.7) and Lemma VI.A.1, the r-th moment of an arbitrary interval of an E(k,p) process is

$$E[(X^{(p)})^r] = \sum_{j=1}^{N} \frac{(r+j-1)!}{(j-1)!} \alpha_j (p\lambda)^{-r} \ . \qquad (7.B.9)$$

Thus,

$$E[X^{(p)}] = (p\lambda)^{-1} \sum_{j=1}^{N} j\alpha_j \quad ,$$

and

$$E[(X^{(p)})^2] = (p\lambda)^{-2} \sum_{j=1}^{N} j(j+1)\alpha_j \quad . \qquad (7.B.10)$$

Recall from Chapter III, Section A that in direct computation of the spectrum of intervals of an E(k,p) process, no provision was made for evaluation of the scale constant $Q_\lambda(k,p)$ (3.A.1.22). Cox and Lewis (1966, pp. 78) relate the initial points of the count and interval spectra of a stationary point process, given in equation (2.A.3.2) and expressed here in the notation of superposition processes,

$$g_+^{(p)}(0+)E[X^{(p)}] = f_+^{(p)}(0+)C^2(X^{(p)}) \quad , \qquad (7.B.11)$$

where $C(X^{(p)})$ is the coefficient of variation of an arbitrary interval of the superposition. Equation (3.C.8) gives the initial value of the E(k,p) count spectrum to be

$$g_+^{(p)}(0+) = p\lambda/(k^2\pi) \qquad (7.B.12)$$

Now (7.B.10), (7.B.11) and (7.B.12) can be used to determine the initial point of the interval spectrum, and thus provide the information necessary to evaluate $Q_\lambda(k,p)$ in terms of the polynomial ratio representation presented in Chapter III.

In Chapter VII, the computational details are described, and pdf's for several values of k and p are displayed which illustrate the results of Theorem VII.A.1, along with other more qualitative features.

VII.C. THE ARBITRARY INTERVAL DENSITY OF THE H(k,p;$\underline{q}$,$\underline{\lambda}$) PROCESS

The component interval density in the H(k,p;$\underline{q}$,$\underline{\lambda}$) process is given by

$$f(t) = \sum_{j=1}^{k} q_j \lambda_j \exp\{-\lambda_j t\}, \qquad (7.C.1)$$

with corresponding survivor and equilibrium excess distributions

$$R(t) = \sum_{j=1}^{k} q_j \exp\{-\lambda_j t\} \qquad (7.C.2)$$

and

$$Z(t) = (1/E[X]) \sum_{j=1}^{k} q_j \exp\{-\lambda_j t\}/\lambda_j , \qquad (7.C.3)$$

where

$$E[X] = \sum_{j=1}^{k} q_j/\lambda_j . \qquad (7.C.4)$$

Using (7.C.1) through (7.C.4) in (7.A.2) gives the density of the arbitrary superposition interval for this process:

$$f^{(p)}(t) = \sum_{j=1}^{k} \sum_{n=1}^{k} q_j q_n \left(\frac{\lambda_j}{\lambda_n} + p - 1 \right) e^{-\lambda_j + \lambda_n)t}$$

$$x \left(\sum_{j=1}^{k} \frac{q_j}{\lambda_j} e^{-\lambda_j t} \right)^{p-2} \Big/ \left(\sum_{j=1}^{k} \frac{q_j}{\lambda_j} \right)^{p-1} . \quad (7.C.5)$$

Inspection of (7.C.5) reveals that $f^{(p)}(t)$ is also hyper-exponential and can be put in the form of

$$f^{(p)}(t) = \sum_{j=1}^{J} \beta_j \mu_j \exp\{-\mu_j t\} ,$$

where

$$J = \binom{k+1}{2} \binom{k+p-3}{p-2} .$$

VIII. NUMERICAL RESULTS

Throughout the course of this research, considerable effort was expended in performing the computational procedures outlined in the various sections of this thesis. The reasons for this effort were

1. To verify the methodology by comparing results of varying approaches to the same problem;
2. To establish the various procedures as practical methods of obtaining answers to particular questions;
3. To provide a basis for comparison between the various methods, and
4. To provide qualitative means of distinguishing the processes studied, and in particular tabulate the forms of the spectra and serial correlations for small p and k.

Chapter III was devoted primarily to deriving an analytic representation for the spectrum of an E(k,p) process. Section A of this chapter describes two methods of evaluating that representation. Included in the section is a brief description of the FORMAC algebraic manipulation preprocessor which has proved useful in several situations.

Chapter V developed a method for computing the spectrum of a semi-Markov generated point process which involves determining the inverse of a matrix $(Ie^{i\omega}-T)$, where ω is the spectral argument. In Section B three methods of solving this problem are presented, each of which has been used successfully in analyzing the interval spectrum of various E(k,p) processes which have been given semi-Markov generated point process representations.

A computational form for the probability density function of an arbitrary interval of an E(k,p) process was derived in Chapter VII. Section C describes implementation of the method, including comparisons of certain specific values with those same values determined through spectral analysis.

Section D contains a brief discussion of the interval bispectra of E(k,p) processes modeled as semi-Markov generated point processes.

Use of the Fast Fourier transform to determine the sequence of serial correlations of a point process from the spectral density is discussed in Section E.

In Section F the computation of the second order count spectrum of an E(k,p) process is presented.

Throughout the chapter we give numerous graphs and tables which illustrate the properties of the functions being studied and provide a basis for comparison between processes.

Computer programs used in this analysis are listed in Appendix E.

VIII.A. DIRECT COMPUTATION OF THE SECOND ORDER E(k,p) SPECTRUM

Lewis, et al (1973, Sect. 5), and Chapter III of this thesis discuss in detail the representation of the interval spectrum of an E(k,p) process as proportional to the sum of the ratios of trigonometric functions (3.A.22), wherein $Q_\lambda(k,p)$ is defined as the constant of proportionality.

Two methods of evaluating (3.A.1.22) have been developed. The first, developed by W. J. Hayne, (See Lewis, et al, 1973)

evaluates the summation numerically for a finite set of values of ω. The second, developed to supplement this research, makes use of the symbolic manipulation capabilities of FORMAC (Raney, 1973) to determine an exact formulation as the ratio of finite polynomials in cos ω, as expected from Theorem III.A.2.

VIII.A.1 Numerical Approximation of the E(k,p) Interval Spectrum

The computational form of the second order spectrum of the E(k,p) process is (3.A.1.22)

$$f_+^{(p)}(\omega) = Q_\lambda(k,p)\,\frac{(1-\cos\,\omega)^p}{\lambda k^{2p}} \sum_{J_1=1}^{k} \cdots \sum_{J_p=1}^{k} \frac{2a_J}{(a_J^2+b_J^2)c_J}\,, \tag{8.A.1.1}$$

with J, a_J, b_J and c_J as defined in (3.A.1.23). A direct numerical evaluation of (8.A.1.1) was implemented in a FORTRAN program, SPECD. The program was used by Lewis, et al. (1973) in the preliminary research of the Erlang superposition process.

The following procedure was followed by SPECD:

a. Enumerate the distinct sets

$$J = \{j_1,\ldots,j_p\},\quad j_i=1,\ldots,k;\ i=1,\ldots,p$$

by setting $j_i=1$, $i=1,\ldots,p$, and incrementing the indices so as to maintain the inequality $j_i \leq j_k$, for $j < k$.

b. For each value of ω chosen uniformly between 0 and π, and each set J, evaluate

$$\frac{2|J|a_J}{(a_J^2+b_J^2)c_J}$$

where $|J|$ represents the number of distinct permutations of the elements of J.

c. For each ω, set

$$U(\omega) = \frac{(1 - \cos \omega)^p}{\lambda k^{2p}} \sum_J \frac{2|J|a_J}{(a_J^2+b_J^2)c_J} \qquad (8.A.1.2)$$

d. Use quadratic extrapolation to evaluate U(o) and $U(\pi)$.

e. Since the integral of $f_+^{(p)}(\omega)$ must equal 1, and $f_+^{(p)}(\omega) = Q_\lambda(k,p)U(\omega)$, approximate $Q_\lambda(k,p)$ by

$$Q_\lambda(k,p) = \{(\pi/N) \sum_{j=0}^{N} U(\omega_j)\}^{-1} , \qquad (8.A.1.3)$$

where $\omega_j = \pi j/N$.

f. Set $f_+^{(p)}(\omega_j) = Q_\lambda(k,p)U(\omega_j)$. (8.A.1.4)

Some additional information available from this program is the variance and coefficient of variation of an arbitrary interval. Using the relation

$$g_+^{(p)}(0+) = p\lambda/(k^2\pi) \qquad (8.A.1.5)$$

and

$$g_+^{(p)}(0+)E[X^{(p)}] = f_+^{(p)}(0+)C^2[X^{(p)}] \ , \qquad (8.A.1.6)$$

previously given as (3.E.6) and (7.B.11), where $X^{(p)}$ represents the interval following an arbitrary event in the superposition process, yields

$$C^2[X^{(p)}] = (f_+^{(p)}(0+)k\pi)^{-1} \ , \qquad (8.A.1.7)$$

and

$$V[X^{(p)}] = \{k/(p^2\lambda^2\pi f_+^{(p)}(0+))\} \ . \qquad (8.A.1.8)$$

Figures VIII.A.1 through VIII.A.10 are comparative spectral density representations of the various E(k,p) spectra generated by the foregoing method.

VIII.A.2 Analytic Determination of the E(k,p) Interval Spectrum

This phase of the research made use of an algebraic manipulation routine, FORMAC, to compute a condensed formula of the second order interval spectrum for specific values of k and p. Theorem III.A.2, guaranteeing that the interval spectrum of an E(k,p) process was that of a mixed auto-regressive/moving average process, required examination with regard to confirming the estimates of the order of the process as proposed in Section III.D.

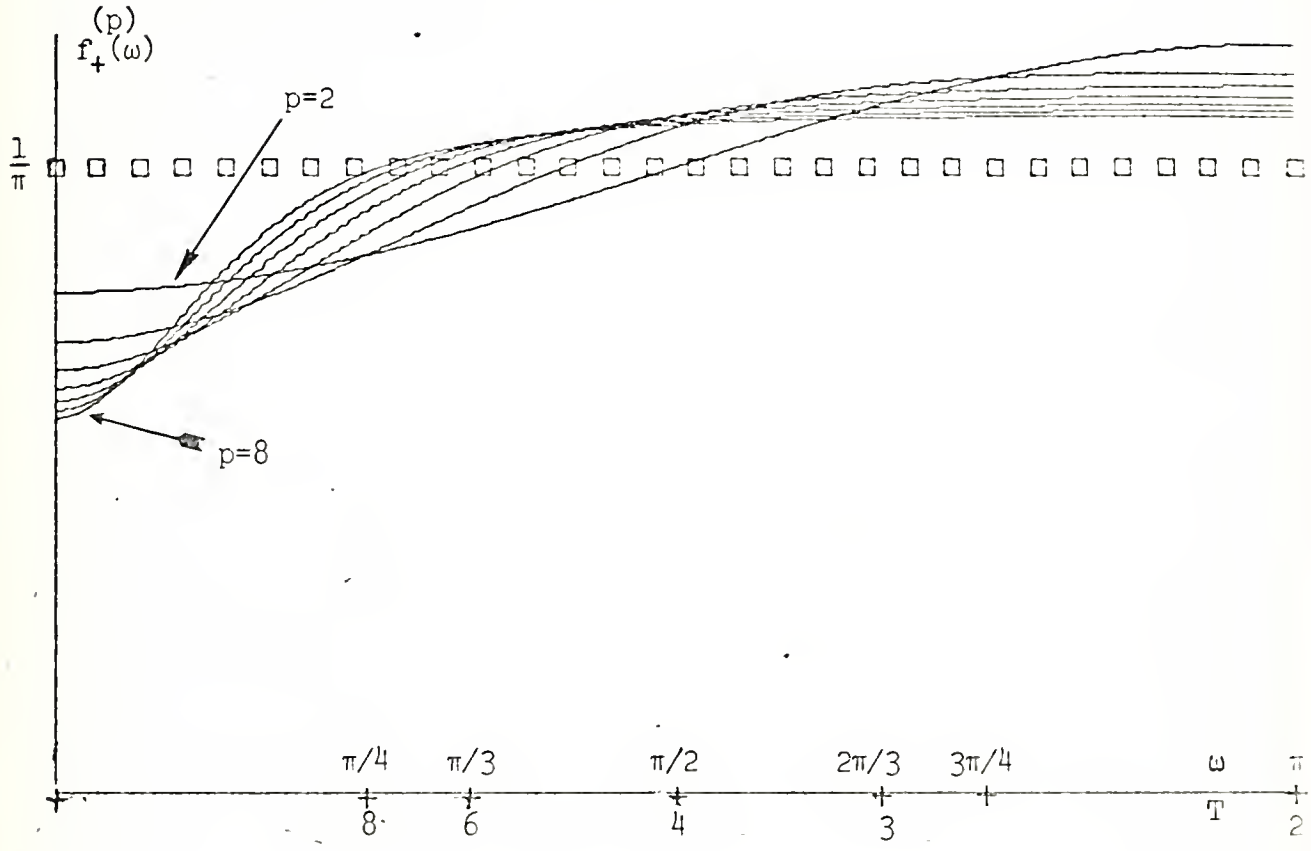


Figure VIII.A.1 Interval spectra, $f_+^{(p)}(\omega)$, for E(2,p) superposition processes, p=2,...,8. The squares at $f_+^{(p)}(\omega) = \frac{1}{\pi}$ represent the flat spectrum of a renewal process. Note that for k=2, the spectrum is fairly flat with a notch developing near the origin with increasing p, as the initial point of the spectrum approaches its asymptotic value, $1/k\pi$. There is a relative peak at $\omega=\pi$ (period T=2) representing the quasi-alternation between events from the two processes for p=2.

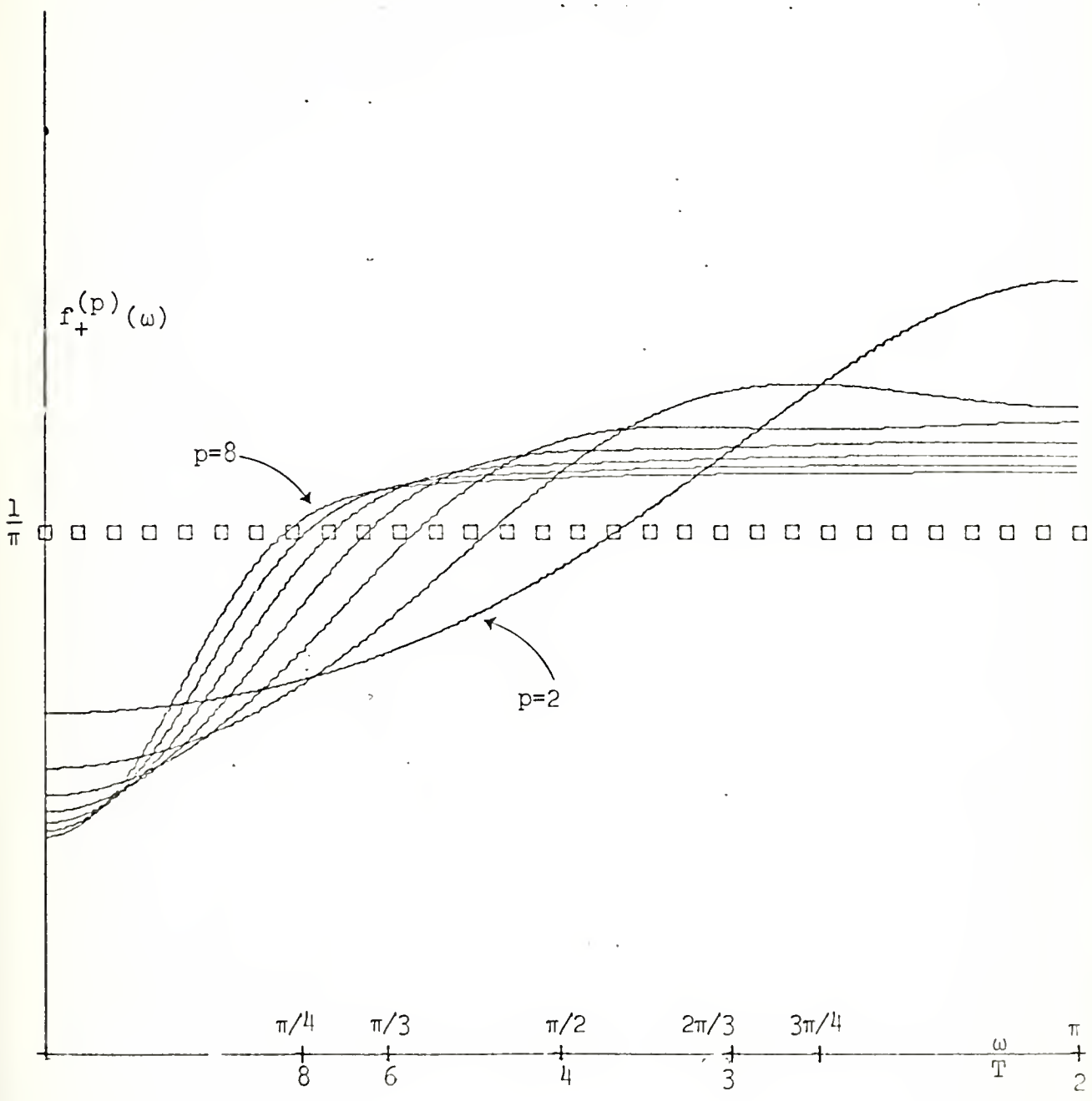


Figure VIII.A.2 Interval spectra for the E(3,p) superposition process, p=2,...,8. The squares of $1/\pi$ represent the flat spectrum of a renewal process. Note the humps in p=2,3 and 4 spectra centered at π(T=2), $2\pi/3$(T=3) and $\pi/2$(T=4) respectively.

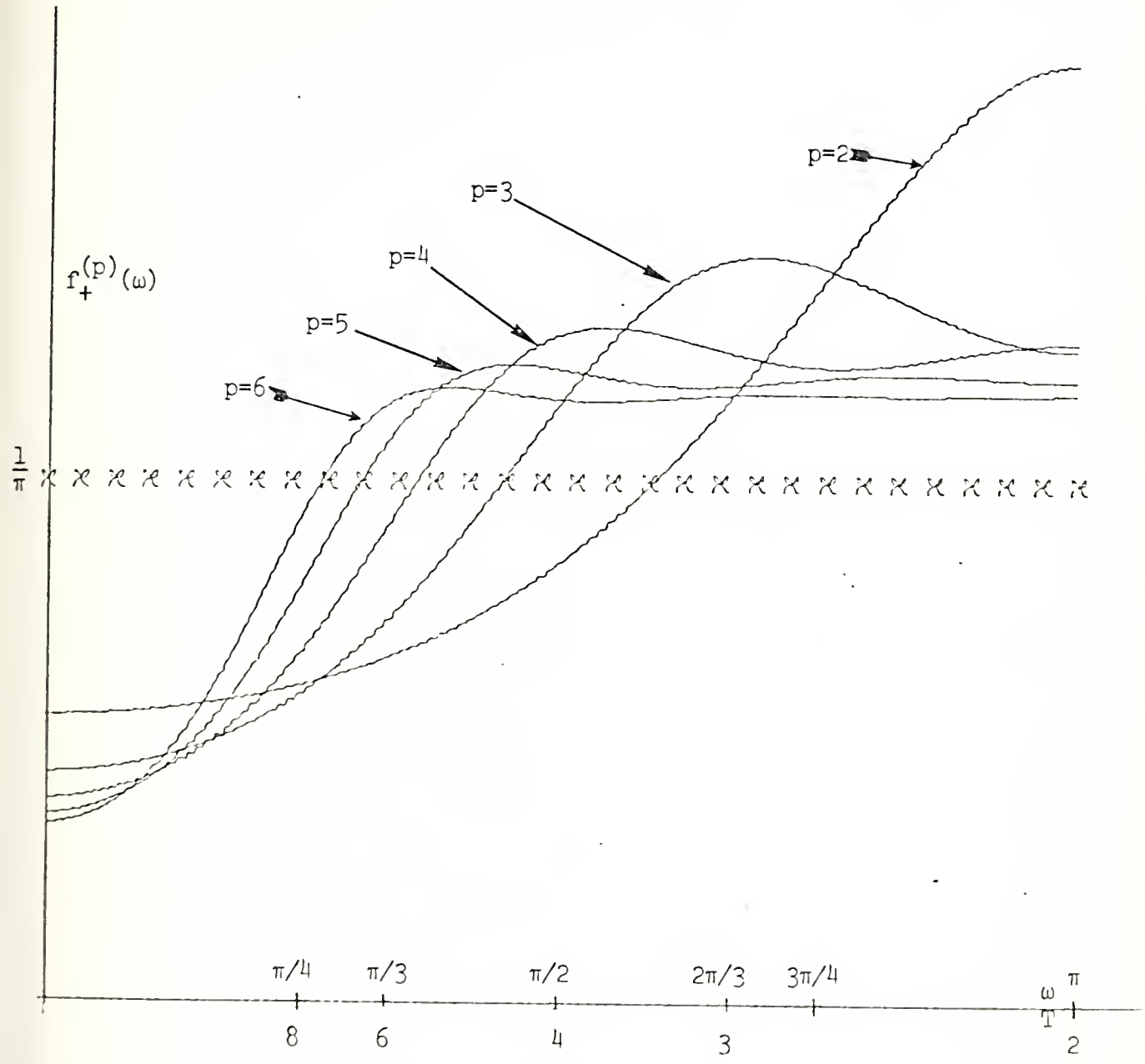


Figure VIII.A.3 Interval spectra for E(4,p) superposition processes, p=2,...,6. The crosses at $f_+^{(p)}(\omega) = \frac{1}{\pi}$ represent the flat spectrum of a renewal process. As in previous cases, relative peaks at periods p appear in the spectrum diminishing in amplitude as p increases and the spectrum approaches the flat spectrum of the Poisson process.

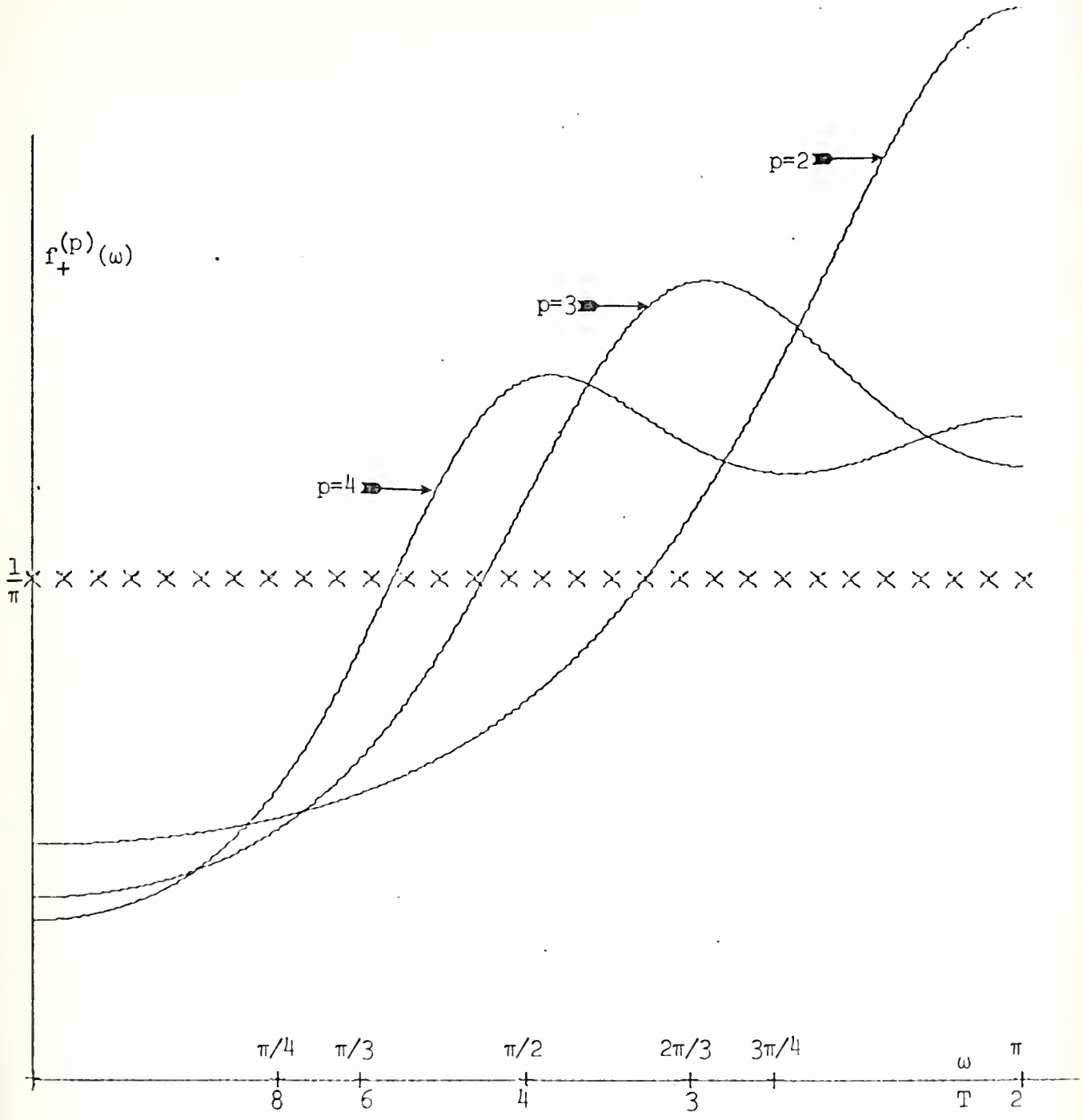


Figure VIII.A.4 Interval spectra for E(5,p) superposition processes, p=2,3,4. The crosses at $f_+^{(p)}(\omega) = \frac{1}{\pi}$ represent the flat spectrum of the renewal process. In addition to the relative peaks exhibited in previous plots, a secondary peak can be seen at T=p/2 for the E(5,4) process. As in previous figures, $f_+^{(p)}(0+)$ decreases monotonically toward $1/k\pi$ as p increases.

Figure VIII.A.5 Interval spectra for the E(6,p) processes, p=2,3,4. The crosses are as in previous figures. As in the E(5,4) process, a second peak at T=p/2 appears in the E(6,p) process, probably representing an harmonic. The main peaks are at periods T=p.

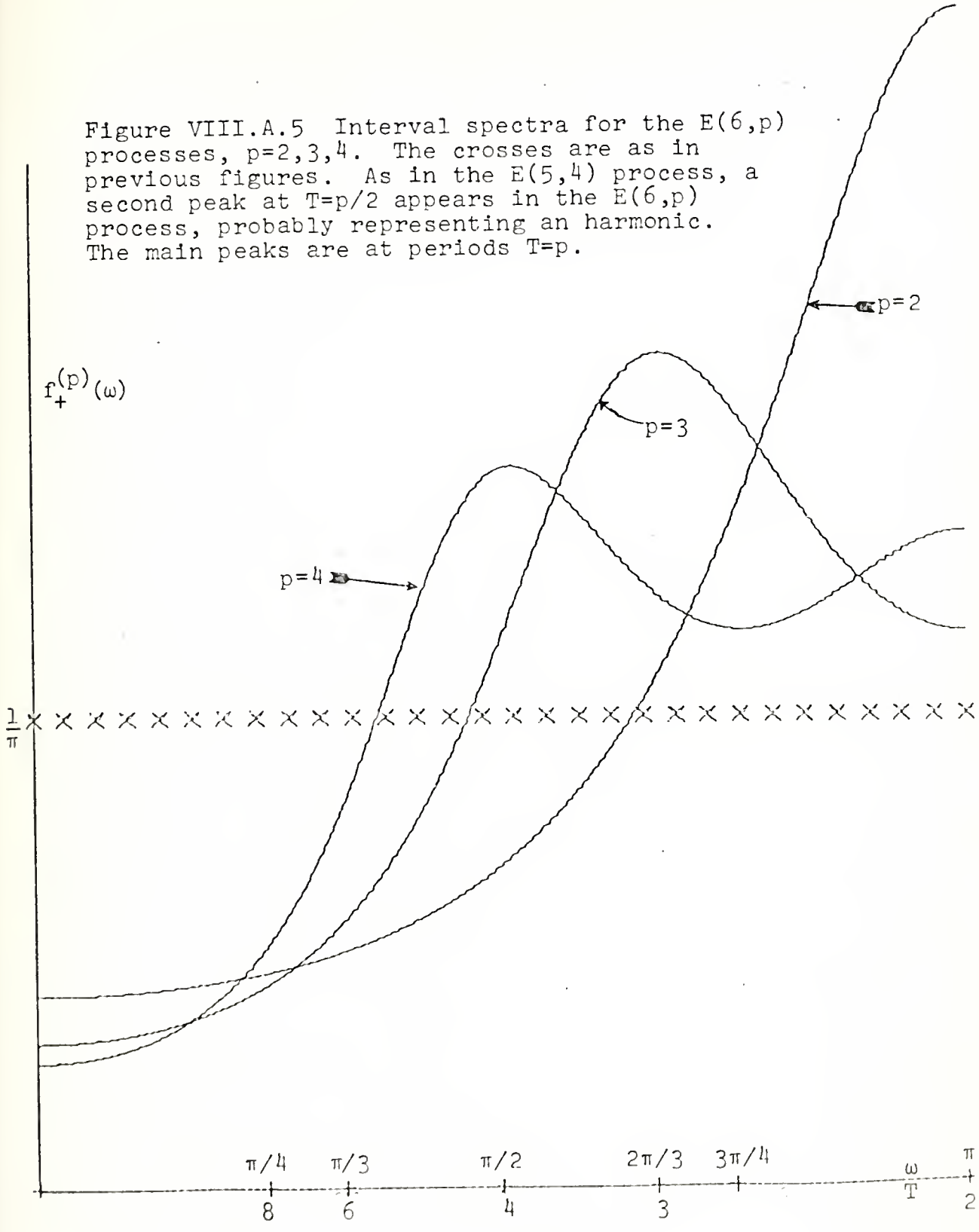
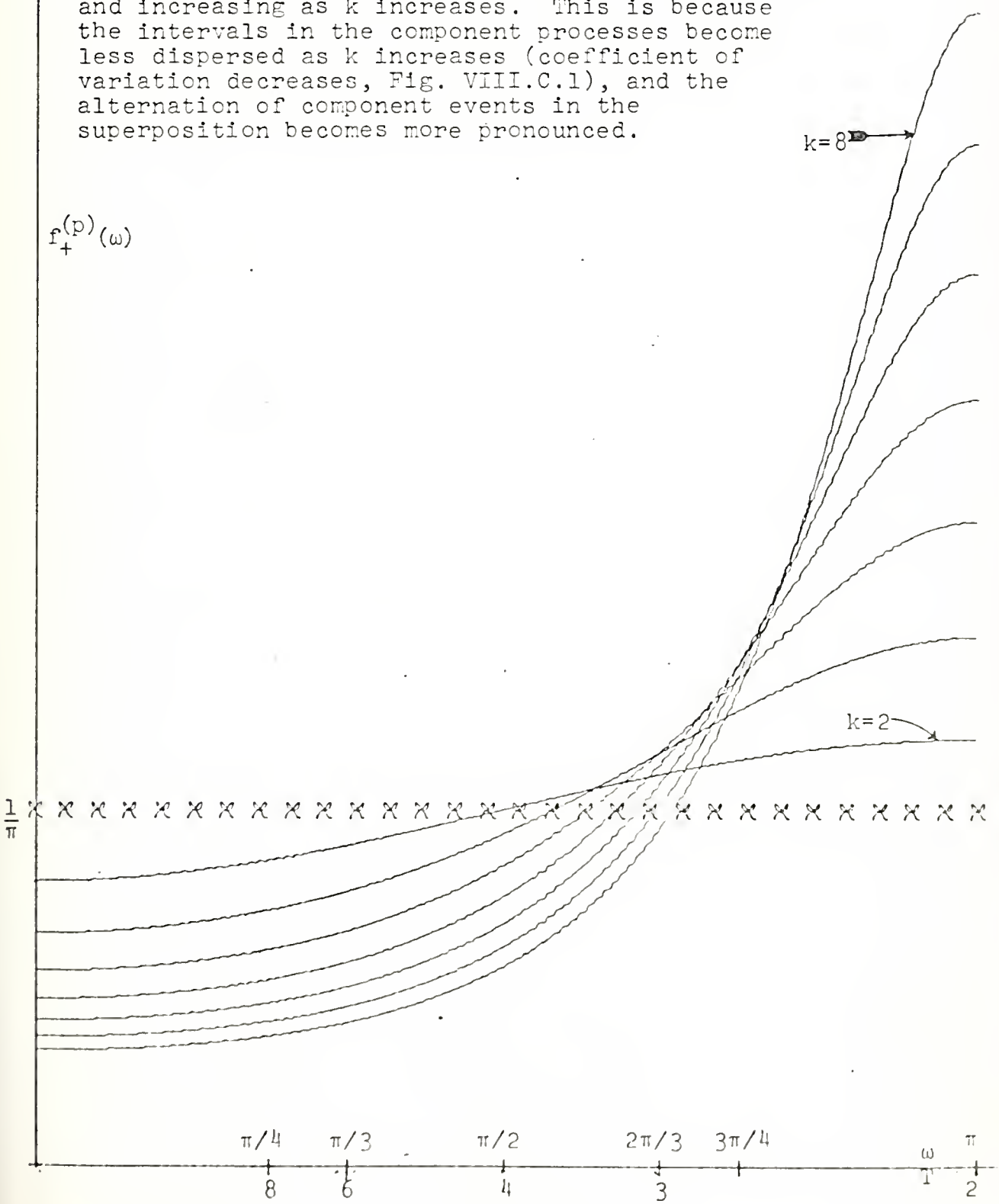


Figure VIII.A.6 Interval spectra for the E(k,2) processes, k=2,...,8. The crosses at $f_+^{(p)}(\omega) = 1/\pi$ are as in previous figures. Note the sharp peak at $\omega=\pi$ representing a period of 2, and increasing as k increases. This is because the intervals in the component processes become less dispersed as k increases (coefficient of variation decreases, Fig. VIII.C.1), and the alternation of component events in the superposition becomes more pronounced.



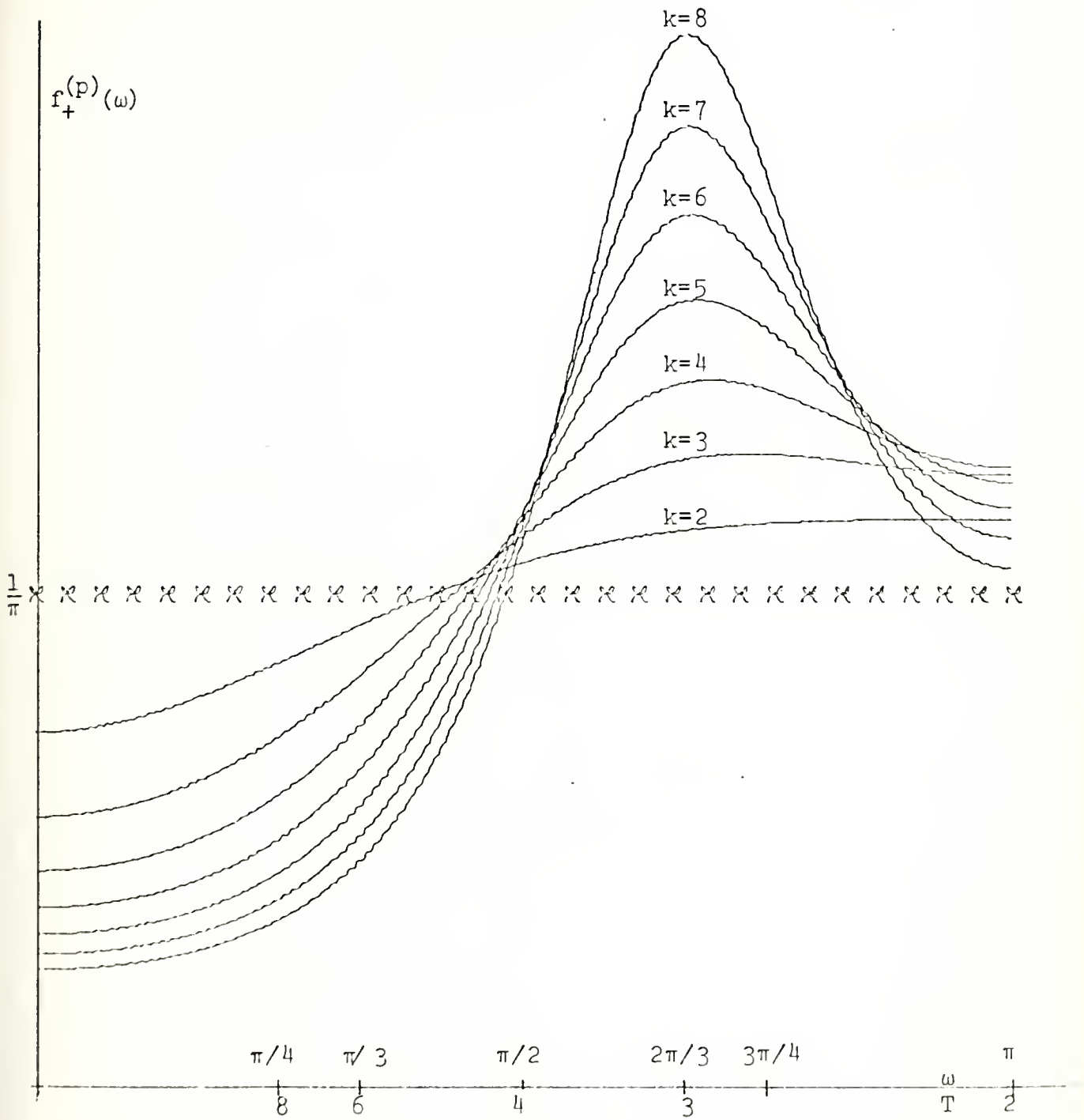


Figure VIII.A.7 Interval spectra for the E(k,3) processes, k=2,...,8. The crosses are as in previous figures. The peaks at $\omega=2\pi/3$ corresponding to a period T=3, clearly are increasing with k.

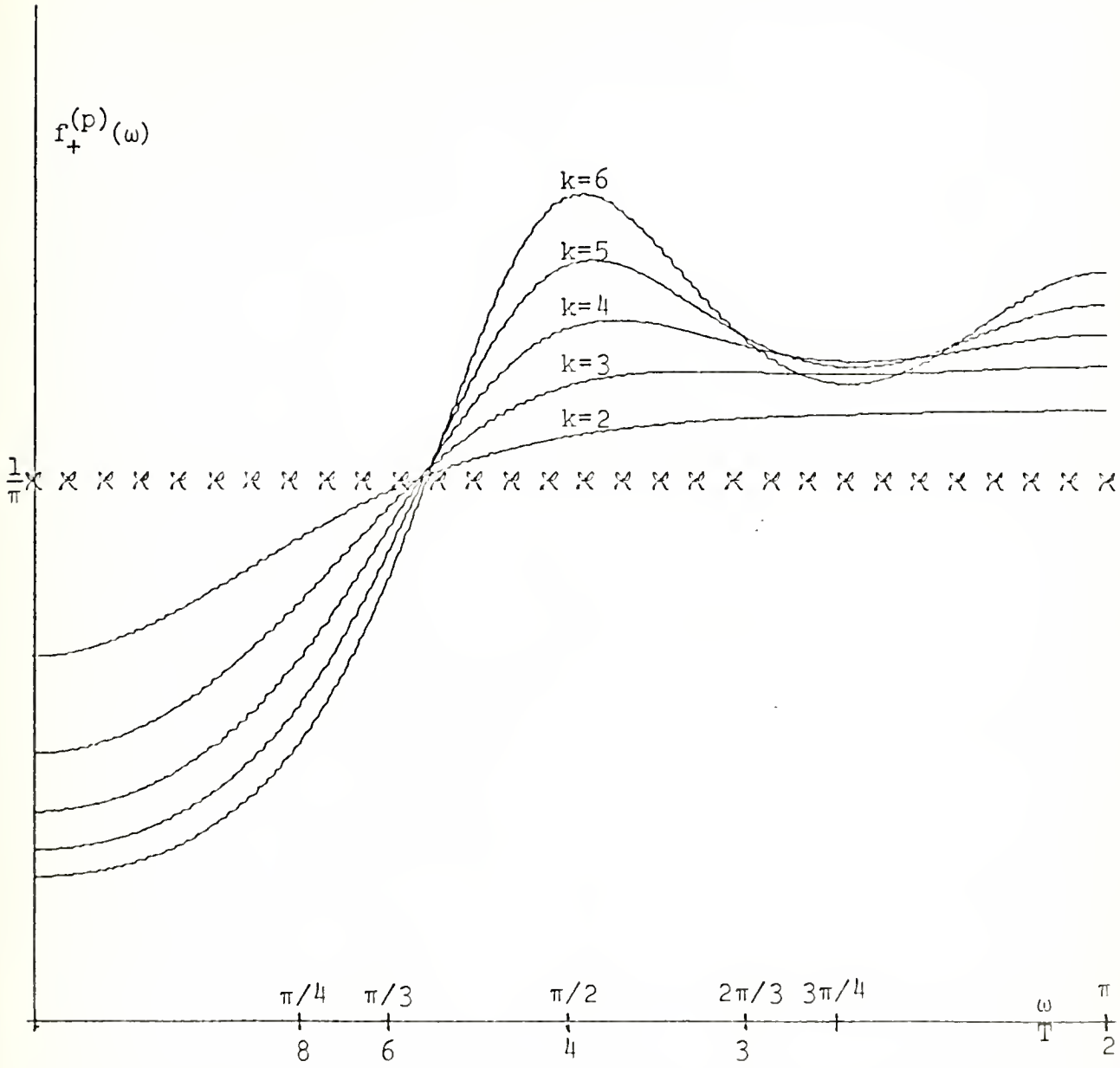


Figure VIII.A.8 Interval spectra for the E(k,4) processes, k=2,...,6. The crosses at 1/π as before. With p even, we see the indication of an harmonic peak at period T=p/2=2, becoming more pronounced as k increases.

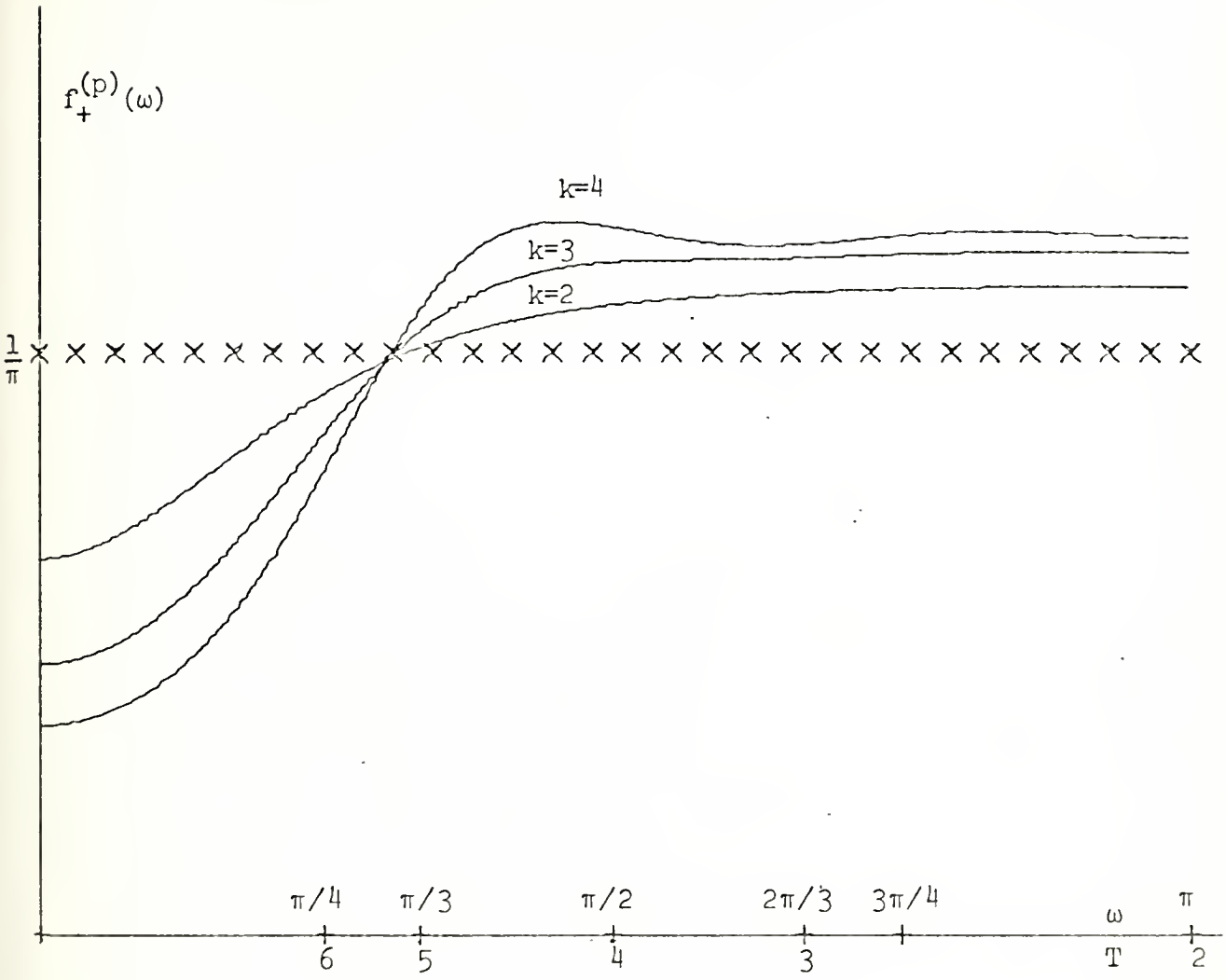


Figure VIII.A.9 Interval spectra of the E(k,5) processes, k=2,3,4. The crosses as before. Here we see a main peak at T=5, $\omega=2\pi/5$, in the E(4,5) process, a secondary peak is apparent at T-5/2, = $4\pi/5$. These peaks would be expected to become more pronounced as k increases.

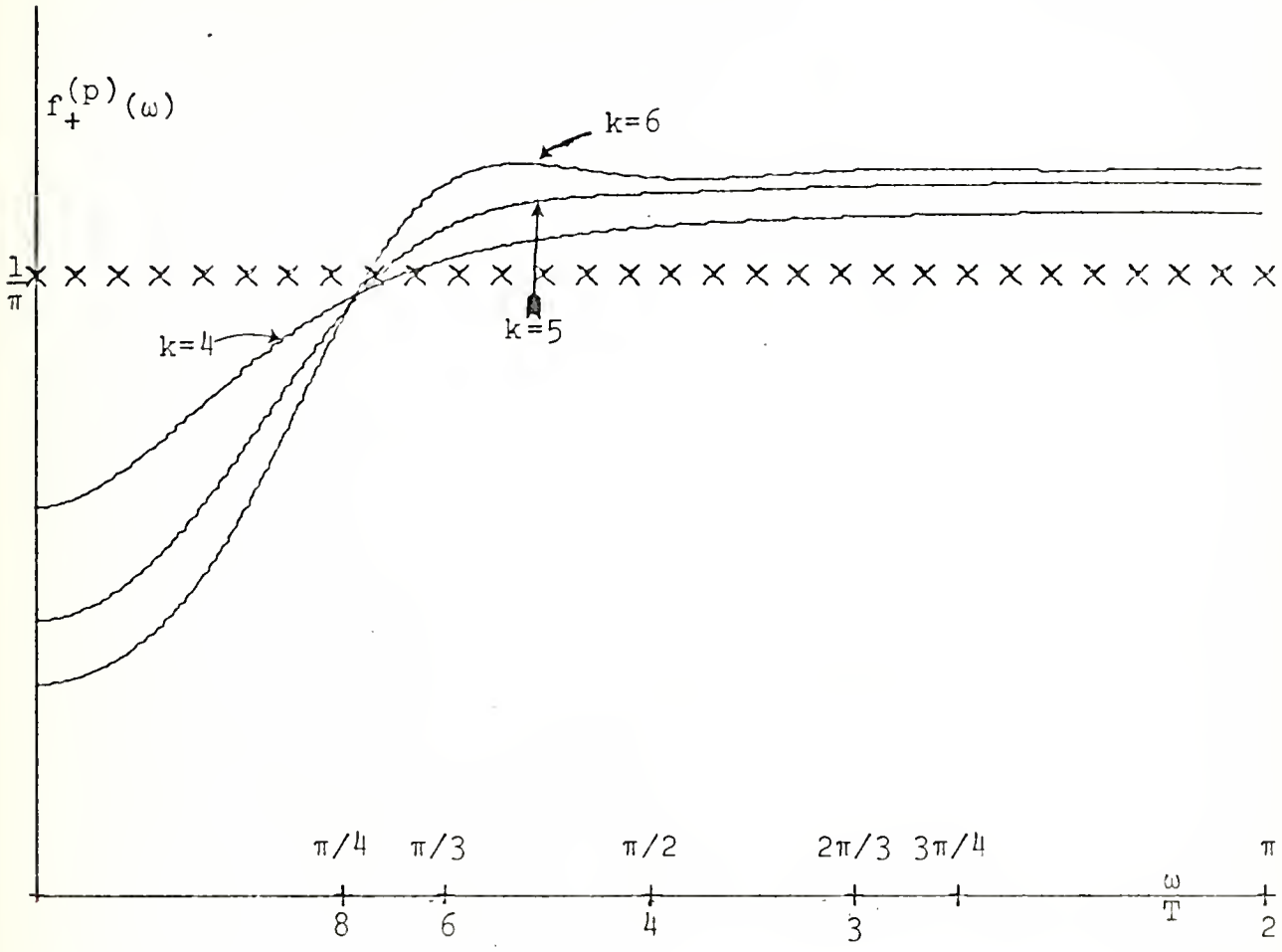


Figure VIII.A.10 Interval spectra for the E(k,6) processes, k=2,3,4. The crosses at $1/\pi$ as before. Here the main peak is at $\omega=\pi/3$ (T=6=p) with a secondary peak at $2\pi/3$(T=p/2) for the E(4,6) process.

While such computation is not difficult to do by hand for k=2, and p=2, 3 or 4 (Fig. VIII.A.11) the magnitude of the coefficients (in integer form) and the number of terms involved for larger values of k and p renders hand calculation virtually impossible. However, integer calculations were required so that differentiation between zeroes and very small numbers would be possible.

Subdivision VIII.A.2.a contains a description of the problem in such a way that an algebraic solution is feasible. In Subdivision VIII.A.2.b wer give a brief description of FORMAC outlining those features which make it suitable for this problem, along with the drawbacks which limit is usefulness.

VIII.A.2.a Problem formulation

To solve algebraically for the interval spectrum of an E(k,p) process, begin with the formulation of Chapter III, Section A, given by equation (3.A.2.4)

$$f_+^{(p)}(\omega) = Q_\lambda(k,p)\,\frac{(1-\cos\omega)^p}{\lambda k^{2p}} \sum_t \binom{p}{R_t} \sum_{L\epsilon R_t} \frac{2a_L}{(a_L^2+b_L^2)c_L} ,$$

where, it will be remembered, R_t is an index designation vector with elements, L_n, which are distinct permutations of the same set of integers. (The sets $V^{(k/2)}(p)$, as defined in Section III.D, are used in place of the index designation vector, R_t, when k is even. The conversion is tedious but straightforward, and discussion will be omitted.)

The general procedure to be followed is to enumerate the index designation vectors, $\{R_t\}$, and form the sum

$$S_t(\omega) = \sum_{L \varepsilon R_t} \frac{2a_L}{(a_L^2+b_L^2)c_L} \qquad (8.A.2.2)$$

for each R_t. Each sum, $S_t(\omega)$, is then placed over a common denominator, so

$$S_t(\omega) = \overline{N}_t(\omega)/\overline{D}_t(\omega).$$

The terms $\overline{N}_t(\omega)/\overline{D}_t(\omega)$ are combined to form

$$S(\omega) = \frac{(1-\cos\ \omega)^p}{\lambda k^{2p}} \sum_t \binom{p}{R_t} \frac{\overline{N}_t(\omega)}{\overline{D}_t(\omega)} \quad , \qquad (8.A.2.3)$$

which again is placed over a common denominator.

Finally, using the known value of $f_+^{(p)}(0+)$, (Equation (7.B.11)) $Q_\lambda(k,p)$ is determined from the equation

$$f_+^{(p)}(0+) = Q_\lambda(k,p)\ S(0+) \quad . \qquad (8.A.2.4)$$

The actual procedure is most easily explained through an example:

Example VIII.A.1: *The interval spectrum of the E(3,3) process.* The class of all index designator sets, R_t, can be operationally defined by

$$\{R_t\} = \{(n_1,\ldots,n_k) \mid p \geq n_j \geq n_{j+1} > 0, j=1,\ldots,k-1; \sum_{j=1}^{k} n_j = p\} \tag{8.A.2.5}$$

For the E(3,3) process

$$\{R_t\} = \{(3,0,0),(2,1,0),(1,1,1)\}$$

$$= \{R_1, R_2, R_3\} .$$

Recalling the definition of $S_t(\omega)$ (8.A.2.2), and the results of Lemma III.A.1, as expressed in (3.A.2.5), (3.A.2.7), (3.A.2.8) and (3.A.2.9),

$$S_1(\omega) = (1/3) \sum_{j=1}^{3} \{1-\cos(\tfrac{2\pi j+\omega}{3})\}^3$$

$$= \frac{(1/3) \sum_{j=1}^{3} [\prod_{m\neq j} \{1-\cos(\tfrac{2\pi m+\omega}{3})\}]^3}{\{2^{-2}(1-\cos\ \omega)\}^p} . \tag{8.A.2.6}$$

At this point a FORMAC program must be written which is tailored to the specific problem. Using the identities expressed in (3.A.2.12) and evaluating the sine and cosine terms, the numerator of $S_1(\omega)$ becomes

$$\begin{aligned}\bar{N}_1(\omega) = {} & [2+\cos(\omega/3) - 3\sin(\omega/3)][1-\cos(\omega/3)](1/2) \\ & + [2+\cos(\omega/3) + 3\sin(\omega/3)][1-\cos(\omega/3)](1/2) \\ & + [2+\cos(\omega/3) - 3\sin(\omega/3)][2+\cos(\omega/3) + 3\sin(\omega/3)]\end{aligned} \tag{8.A.2.7}$$

which is easily evaluated by FORMAC. Substituting the appropriate identities from Appendix B results in

$$S_1(\omega) = \frac{3(139 + 100\cos\omega + 4\cos^2\omega)}{(1-\cos\omega)^3} \qquad (8.A.2.8)$$

Similarly,

$$S_2(\omega) = \sum_{j=1}^{3} \sum_{m\neq j} \frac{(3-2\cos\frac{2\pi j+\omega}{3} - \cos\frac{2\pi m+\omega}{3})(1-\cos\frac{2\pi j+\omega}{3})^{-2}(1-\cos\frac{2\pi m+\omega}{3})^{-2}}{(7-6\cos\frac{2\pi j+\omega}{3} - 3\cos\frac{2\pi j+\omega}{3} + 2\cos\frac{2\pi(m+j)+\omega}{3})} \qquad (8.A.2.9)$$

is evaluated in a specially constructed FORMAC program, using appropriate trigonometric identities, and becomes

$$S_2(\omega) = \frac{4(14274 + 2439\cos\omega - 918\cos^2\omega + 81\cos^3\omega)}{(1-\cos\omega)^2(27\cos^2\omega+169)} \qquad (8.A.2.10)$$

The index designator set R_3 represents a single element which, when evaluated as above gives

$$S_3(\omega) = \frac{8}{3(1-\cos\omega)} \qquad (8.A.2.11)$$

Substituting (8.A.2.8), (8.A.2.9) and (8.A.2.11) in (8.A.2.3) yields the final form of $S(\omega)$:

$$S(\omega) = (1-\cos\omega)^3\{S_1(\omega)+3S_2(\omega)+6S_3(\omega)\}/\lambda 3^6$$

$$= \frac{(249873+142020\cos\omega-22389\cos^2\omega+17496\cos^3\omega+648\cos^4\omega)}{(169+27\cos^2\omega)\lambda} \qquad (8.A.2.12)$$

Using the results of Chapter VII (7.B.10) and (7.B.12) as implemented in Section B of this chapter yields

$$g_+^{(3)}(0+) = \lambda/(3\pi) ,$$

$$c^2[X^{(3)}] = \frac{3983}{6561} ,$$

and

$$E[X^{(3)}] = 1/\lambda . \qquad (8.A.2.13)$$

Using (8.A.1.6) and (8.A.2.13) gives the result

$$f_+^{(3)}(0+) = (\lambda/3\pi)(1/\lambda)/(3983/6561)$$

$$= \frac{6561}{(3983)(3)} . \qquad (8.A.2.14)$$

Evaluating (8.A.2.12) at $\omega=0$ and using it along with (8.A.2.14), gives the scale factor $Q_\lambda(3,3)$:

$$Q_\lambda(3,3) = \frac{3\lambda}{3983\pi} \qquad (8.A.2.15)$$

Thus the final expression for the E(3,3) interval spectrum becomes

$$f_+^{(3)}(\omega) = \frac{3(249873+142020\cos\omega-22389\cos^2\omega+17496\cos^3\omega+648\cos^4\omega}{(169+27\cos^2\omega)3983\pi} \qquad (8.A.2.16)$$

This completes Example VIII.A.1. However reference will be made to this example in the following section. Exact results for processes evaluated in this manner are tabulated in Figure VIII.A.11.

Recall that the order of an ARMA(m,n) process is reflected in the degree of the cosine polynomials in the spectral representation. In the case of the E(3,3) process, the similar process is an ARMA(2,4) process, giving a spectrum whose numerator is a cosine polynomial of degree 4 and whose denominator is a cosine polynomial of degree 2. Note that not every ARMA(2,4) process is similar to the E(3,3) process; only the one whose spectrum is given by (8.A.2.16).

An important feature of the results tabulated in Fig. VIII.A.13 is their consistency with the degree of the similar ARMA processes tabulated in Figure III.C.1.

VIII.A.2.b. The FORMAC algegraic manipulation preprocessor

> "FORMAC (FORmula MAnipulation Compiler) provides...a tool for performing symbolic manipulation of mathematical expressions along with the usual features of a numerical mathematical language... Expressions to be manipulated can contain variables, user-defined functions, constants, symbolic constants and functions. Expressions can be differentiated, evaluated, replaced, compared and parsed."[1]

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[1] Raney (1973, pp. 1).

Figure VIII.A.11 Algebraic representations as ratios of cos ω polynomials of the second order interval spectra of certain Erlang superposition processes. The expressions given for each spectrum are denoted N, D and Q. The spectrum is

$$f_+(\omega) = QN/D$$

Those processes marked with an asterisk were computed by Lewis, et al. (1973).

The computational difficulty associated with this computation is illustrated by the resultant formula for the E(4,4) interval spectrum.

These were computed by hand for small k and p, and with FORMAC assistance for higher k and p.

E(2,2) process* {ARMA(1,0) spectrum}

$N = 1 - \cos \omega$

$D = 1$

$Q = 1/5$

E(2,3) process* {ARMA(2,1) spectrum}

$N = 121 - 48 \cos \omega - 9 \cos^2 \omega$

$D = 41 - 9 \cos \omega$

$Q = 9/25$

E(2,4) process* {ARMA(3,1) spectrum}

$N = 871 - 502 \cos \omega - 57 \cos^2 \omega - 24 \cos^3 \omega$

$D = 17 - 8 \cos \omega$

$Q = 8/379$

E(3,2) process {ARMA(2,1) spectrum}

$N = 3942 + 2241 \cos \omega + 378 \cos^2 \omega$

$D = 65 + 16 \cos \omega$

$Q = 4/495$

Figure VIII.A.11 (continued)

E(3,3) process {ARMA(4,2) spectrum}

$$N = 249,873 + 142,020 \cos \omega - 22,389 \cos^2\omega + 17,496 \cos^3\omega + 648 \cos^4\omega$$

$$D = 169 + 27 \cos^2\omega$$

$$Q = 3/3983$$

E(4,2) process {ARMA(2,1) spectrum}

$$N = 31 - \cos^2\omega$$

$$D = 3 + \cos \omega$$

$$Q = 256/3495$$

E(4,3) process {ARMA(5,3) spectrum}

$$N = 182,141,315,652,508 - 78,862,095,096,808 \cos \omega - 42,822,137,630,712 \cos^2\omega + 25,599,977,898,816 \cos^3\omega + 207,765,000 \cos^4\omega$$

$$D = 43,835,592,913 - 19,552,361,549 \cos \omega + 19,683,026,163 \cos^2\omega - 4,100,625 \cos^3\omega$$

$$Q = 6561/8,913,904$$

E(4,4) process {ARMA(8,5) spectrum}

$$N = 310,464,417,651,734,009/8 + (1/8)11,776,887,383,690,599,849 \cos \omega + (1/8)25,218,699,913,699,885,013 \cos^2\omega + (1/8)21,251,879,278,015,791,883 \cos^3\omega - 1,060,176,380,434,254,459 \cos^4\omega + 426,910,575,503,195,072 \cos^5\omega + 544,710,307,072,237,568 \cos^6\omega + 17,367,184,018,767,872 \cos^7\omega - 1,090,519,040,000 \cos^8\omega$$

$$D = 230,024,100,214,001 - 64,640,820,663,704 \cos \omega + 115,164,588,505,856 \cos^2\omega + 75,974,636,240,896 \cos^3\omega - 8,481,012,711,724 \cos^4\omega - 335,554,320,000 \cos^5\omega$$

$$Q = 1,556,391,325,136 \times 10^{10}/833,625,450,472,186,167,743,918,509$$

Figure VIII.A.11 (continued)

E(5,2) process {ARMA(3,2) spectrum}

$N = 4{,}043{,}205 - 47{,}553 \cos \omega - 224{,}126 \cos^2 \omega + 9{,}728 \cos^3 \omega$

$D = 1{,}174{,}529 + 720{,}192 \cos \omega - 4{,}096 \cos^2 \omega$

$Q = 160/679$

The above description summarizes the properties of FORMAC. The specific capabilities of FORMAC which loaned themselves to this phase of the research were

1.) Ability to maintain formula integrity. That is, to keep track of variables in symbolic form so that the contribution of each parameter to the final result is directly apparent.

2.) Ability to perform polynomial multiplication.

3.) The capacity to maintain extensive formulae in exact form. Integer coefficients can have up to 2295 digits, while formula length is only limited by core storage available to the program.

4.) The capability of implementing simplifying substitutions, as for example, the trigonometric identities of Appendix B.

In evaluating expressions of the form shown in (8.A.2.9) the identity

$$\cos\left(\frac{2\pi j+\omega}{k}\right) = \cos(2\pi j/k)\cos(\omega/k) - \sin(2\pi j/k)\sin(\omega/k)$$

was used to isolate the variable terms $\cos(\omega/k)$ and $\sin(\omega/k)$. For many values of j and k, there are no rational equivalents to $\cos(2\pi j/k)$ and $\sin(2\pi j/k)$. Thus, to ensure that cancellation of terms, where it occurred, would not be masked by numerical imprecision, and conversely, not be indicated by very small coefficients, the terms $\cos(2\pi j/k)$ and $\sin(2\pi j/k)$ were retained as symbolic entities.

Equations (8.A.2.9) and (8.A.2.12) illustrate the need for polynomial multiplication. There $S_2(\omega)$ is the sum of polynomial fractions in $\sin(\omega/3)$ and $\cos(\omega/3)$. Each step of the procedure which expresses $S_2(\omega)$ as a single

polynomial fraction requires multiplication of polynomials and collection of like terms. At the same time, these polynomial products involve many terms in various combinations of

$$\sin^r(2\pi j/k)\cos^s(2\pi m/k)\cos^t(\omega/k)\sin^v(\omega/k) \qquad (8.A.2.17)$$

which must be maintained in their entirety until simplifying relations can be imposed.

The method of generating the compact form of the spectrum of an E(k,p) process is iterative in nature. It is not clear at the outset which trigonometric identities will reduce the complicated formulae to the desired form. The problem must first be run without substitutions of any kind. Based on the results of the preliminary run, a set of substitutions designed to convert $\sin^{2v}(\omega/k)$ to $(1-\cos^2(\omega/k))^v$ and all symbolic coefficients of $\cos^t(\omega/k)$ and $\sin^{2v-1}(\omega/k)$ to rational numbers. (It is not clear that this is always possible, but it was worked out for those cases actually considered.) The final stage requires the conversion of $\cos^t(\omega/k)$ to $\cos(t\omega/k)$. At this step, correct formulation, correct substitution and Theorem III.A.2 guarantee the cancellation of terms for which t is not an integral multiple of k.

Two serious drawbacks to this procedure exist. First, the FORMAC analysis is extremely demanding in terms

of computer space and time. A final run for the spectrum of the E(5,2) process, once the appropriate substitutions had been determined, required 378000 bytes of storage and 73 minutes of central processing unit time on the IBM 360/67 computing system.

The second drawback is in determining the first stage substitution requirements and substitution identities. As can be seen in (8.A.2.17), a great many combinations are possible. In the E(5,2) analysis, with

$$a_1 = \cos(2\pi/5) = -\cos(3\pi/5) = -\cos(7\pi/5) = \cos(8\pi/5)$$

$$b_1 = \sin(2\pi/5) = \sin(3\pi/5) = -\sin(7\pi/5) = -\sin(8\pi/5)$$

$$a_2 = \cos(4\pi/5) = \cos(6\pi/5) = -\cos(9\pi/5) = -\cos(\pi/5)$$

$$b_2 = \sin(4\pi/5) = -\sin(6\pi/5) = -\sin(9\pi/5) = \sin(\pi/5)$$

some typical substitutions were

$$b_1^3 b_2 a_2^3 a_1 - b_2^3 b_1 a_1^3 a_2 = -5/2^6$$

$$a_1^2 + a_2^2 = 3/2^2$$

$$(a_1 a_2 b_1 b_2)^2 = 5/2^8 \quad .$$

All of the above forms were derived from the basic identities

$$a_1 + a_2 = -1/2 \quad ,$$

and

$$a_1 = 2a_2^2 - 1 = 1 - 2b_2^2 \quad .$$

In all, 41 substitutions were required for the second stage computations. The number of such substitutions increases rapidly with k and p, along with the complexity of the terms, the time and the space required for the computations. Using this method on a process such as the E(7,3) process is virtually impossible.

VIII.B. THE INTERVAL SPECTRUM OF THE SEMI-MARKOV GENERATED POINT PROCESS

The interval spectrum of a semi-Markov generated point process, as shown in Chapter V, (5.B.5) involves the inverse of the matrix (Iu-T). In order to form a workable expression for the spectrum, a practical method for performing the inversion of a matrix containing a variable term is required. Three methods have been investigated for this purpose. The methods are designated algebraic inversion, diagonalization and numerical approximation.

Subsection VIII.B.1 details the method of algebraic inversion, by which a modified Gauss elimination written for the FORMAC preprocessor was used to find $(Iu-T)^{-1}$ directly.

Subsection VIII.B.2 describes the use of diagonalizing matrices P and P^{-1} to find

$$(Iu-T)^{-1} = P(Iu-D)^{-1}P^{-1}$$

where $D=P^{-1}TP$ is a diagonal matrix.

Subsection VIII.B.3 contains an explanation of the use of equation (5.B.12) to generate a numerical approximation of the interval spectrum. In this method, the summation

$$\sum_{j=1}^{\infty} \underline{\pi}^{T} M_1 (T^{j-1} - T^{\infty}) M_1 \underline{1} \cos j\omega$$

$$\tilde{=} \sum_{j=1}^{N} \underline{\pi}^{T} M_1 (T^{j-1} - T^{\infty}) M_1 \underline{1} \cos j\omega$$

with the error decreasing geometrically with N.

VIII.B.1 Algebraic Inversion of (Iu-T)

The method of direct algebraic inversion relies on the theory of integer preserving Gaussian elimination. (See for example Bareiss (1968))

The procedure is this:

To the matrix (Iu-T), and an identity matrix of the same dimension, apply a sequence of identical transformations which will transform (Iu-T) to an identity matrix. Then the matrix which results from the sequence of transformations applied to the identity matrix will be $(Iu-T)^{-1}$. In the iterative Gaussian procedure, denote the state of the original matrix after it has been operated upon n-1 times by $\{a_{ij}^{(n-1)}\}$, and the correspondingly transformed identity matrix by $\{b_{ij}^{(n-1)}\}$.

Let $d^{(n)} = a_{nn}^{(n-1)}$, $n=1,\ldots,m$; $d^{(0)}=1$, where m is the dimension of the matrix T. Then for $i \neq n$,

$$a_{ij}^{(n)} = (a_{ij}^{(n-1)} d^{(n)} - a_{nj}^{(n-1)} a_{in}^{(n-1)})/d^{(n-1)} \qquad (8.B.1.1)$$

$$b_{ij}^{(n)} = (b_{ij}^{(n-1)} d^{(n)} - b_{nj}^{(n-1)} a_{in}^{(n-1)})/d^{(n-1)} \qquad (8.B.1.2)$$

and

$$a_{nj}^{(n)} = a_{nj}^{(n-1)} \quad ; \quad b_{nj}^{(n)} = b_{nj}^{(n-1)} \qquad (8.B.1.3)$$

Two important features of this computation are, from Fox (1965, Ch. 3),

a. The term $d^{(n-1)}$ divides the terms of (8.B.1.1) and (8.B.1.2) exactly, and

b. If (Iu-T) is of order m, then $d^{(m)}$ is the determinant of (Iu-T), and $\{b_{ij}^{(m)}\}$ is the adjoint. That is, if $B=\{b_{ij}^{(m)}\}$, then

$$(Iu-T)^{-1} = B/d^{(m)} \qquad (8.B.1.4)$$

Although the divisibility property, a, is easily verified, it must be remembered that $d^{(n)}$ is a polynomial in u for n=1,...,m. Initially, we have $d^{(1)}=u-t_{11}$, and from (8.B.1.1),

$$d^{(2)} = \{(u-t_{22})(u-t_{11})-t_{12}t_{21}\}/1 \quad .$$

Similarly, the terms $a_{ij}^{(n)}$ and $b_{ij}^{(n)}$ are polynomials in u. It is necessary, therefore to perform polynomial division to force each new element into the proper form.

A polynomial division algorithm presented by Knuth (1969) was used: Let $A=a_0x^n+\ldots+a_{n-1}x+a_n$, and let $B=b_0x^m+\ldots+b_{m-1}x+b_m$, with $m\leq n$. It is desired to find $C=c_0c^k+\ldots+c_{k-1}x+c_k=A/B$. The result requires n-m+1 iterations with k=n-m. At iteration j=0,1,...,k,

$$c_j = a_j^{(j)}/b_0$$

where

$$a_r^{(j)} = \begin{cases} a_r^{(j-1)} - c_j b_r \ , & r=j,j+1,\ldots,j+m \\ a_r^{(j-1)} \ , & r>j+m \end{cases}$$

and

$$a_r^{(0)} = a_r \ , \ r=0,\ldots,n \ .$$

This scheme was implemented using the FORMAC preprocessor. The special capabilities of FORMAC which suited it to this problem were, in additon to those previously listed, the expression analysis capabilities. In particular,

a. The ability to extract coefficients of given terms from a given expression, and

b. The ability to determine the greatest and lowest powers of a given term.

At any point of the inversion that a division A/B was required, FORMAC commands determined the degree of A

and B as polynomials in u, extracted the coefficients of u^j in both expressions, and executed the division algorithm.

To evaluate this and the other methods listed below, several semi-Markov point processes were constructed which were equivalent to the Erlang and hyperexponential superpositions using the methods detailed in Chapter VI. Of interest here are the compact formula for the interval spectra as compared to those tabulated in Figure VIII.A.11, and the plots of the hyperexponential superposition spectra shown in Figures VIII.B.1 and VIII.B.2. Plots of the Erlang superposition spectra from this method are not shown as they are identical for small k and p with those produced by the numerical routine, SPECD, which is considerably more efficient in terms of both time and storage requirements.

It should be noted at this point that producing the exact polynomial form of the E(5,2) process interval spectrum required less than two minutes by the algebraic inversion method, as distinguished from the 73 minute run using the direct computations of Subsection VIII.A.2. As the dimension of the transition matrices increases, numerical error seriously interferes with the results.

VIII.B.2. The Diagonalization Method

In this method of matrix inversion, it is necessary to find a matrix P such that TP=PD, where D is a diagonal matrix.[1] In particular, the matrix D has the eigenvalues

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[1]This method was suggested by Dr. M. Rosenblatt during a visit to the Naval Postgraduate School.

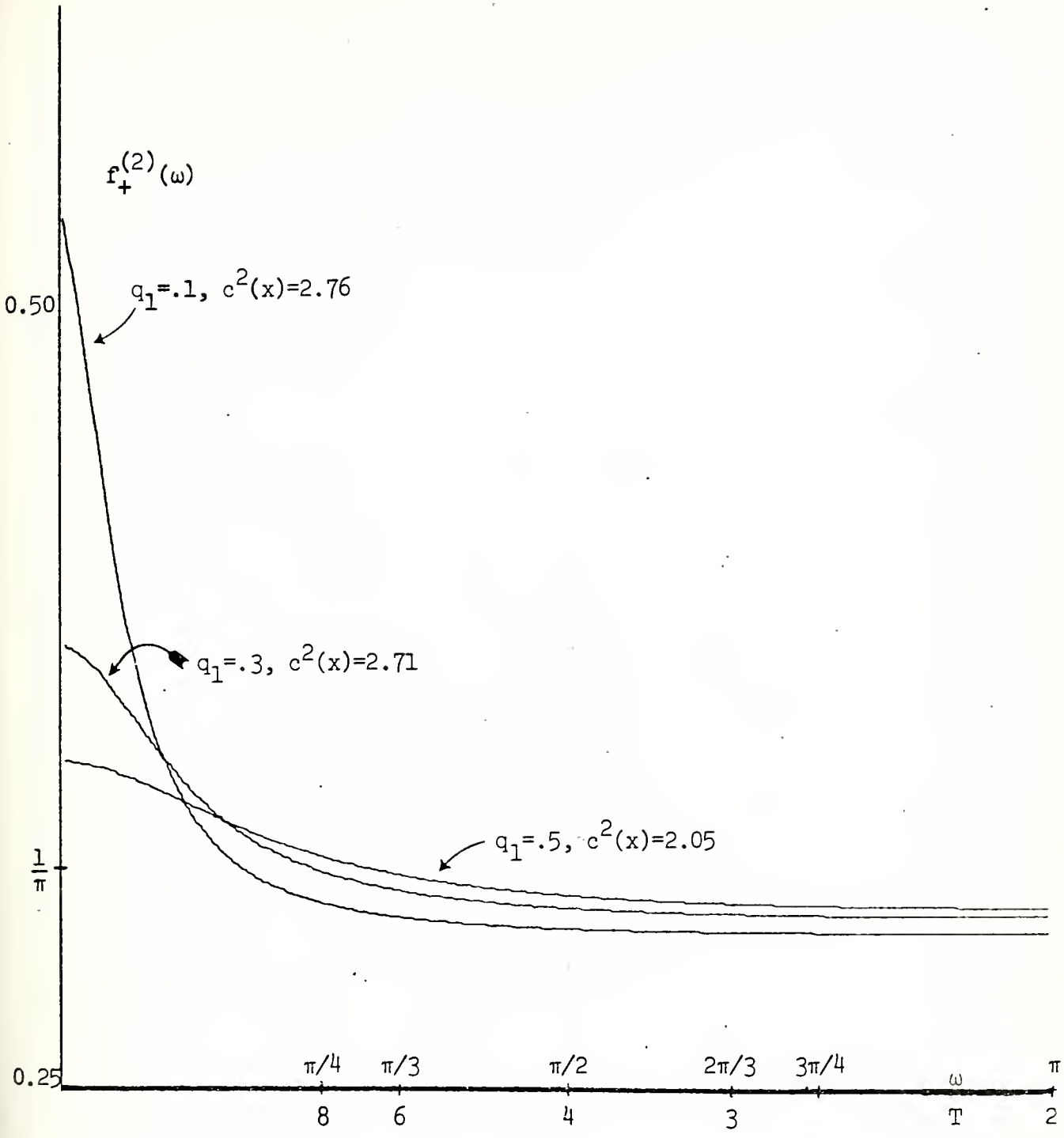

Figure VIII.B.1 Interval spectra for the H(2,2;$\underline{q}$,$\underline{\lambda}$) superposition process; λ_1=.1, q_1=.1,.3,.5; λ_2=1-λ_1, q_2=1-q_1. These spectra were obtained from the semi-Markov generated point process representation of the H(2,2;$\underline{q}$,$\underline{\lambda}$) process.. Note that the sharpest peak results from the process in which the dominant exponential component has the highest intensity. The coefficient of variation refers to the superposition interval.

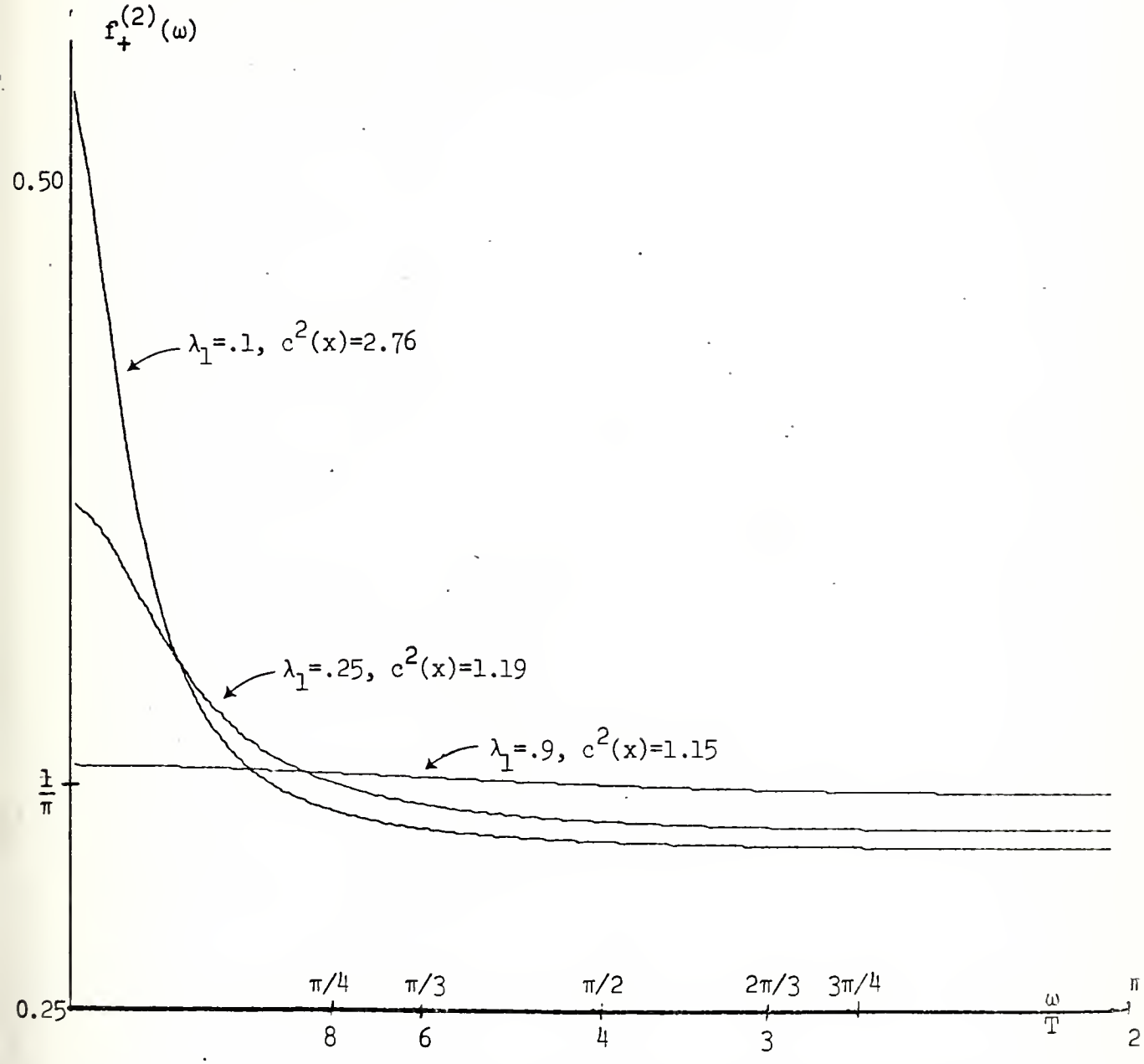

Figure VIII.B.2 Interval spectra for the H(2,2;$\underline{q}$,$\underline{\lambda}$) superposition process q_1=.1, λ_1=.1,.25,.9, λ_2=1-λ_1. Construction of this graph as in Figure VII.B.1. Note that with the dominant component in the hyperexponential distribution having a slow rate, λ_2=.1 the spectrum is nearly flat. Comparison with Fig. VIII.B.1 indicates that the spectral density is not closely related to the coefficient of variation.

of the matrix T as its diagonal elements. The matrix P may be made up of linearly independent representations of the right eigenvectors associated with the eigenvalues of T. Then

$$T = PDP^{-1}$$

so that

$$(Iu-T)^{-1} = (PP^{-1}u-PDP^{-1})^{-1}$$

$$= P(Iu-D)^{-1}P^{-1} \quad . \qquad (8.B.2.1)$$

Clearly, $(Iu-D)$ is a diagonal matrix with elements $u-\lambda_i$, where $\lambda_i = d_{ii}$ is the i-th eigenvalue of T, so the inverse can be written directly as

$$(Iu-D)^{-1} = \{\delta_{ij}(u-\lambda_i)^{-1}\} \qquad (8.B.2.2)$$

where δ_{ij} is the Kroneker delta.

Referrring to (5.B.5),

$$f_+(\omega) = \underline{\pi}^T\{M_2+M_1[(Iu(\omega)-T)^{-1}+(Iu(-\omega)-T)^{-1}]M_1\}\underline{1}$$

$$= \underline{\pi}^T\{M_2+M_1P[(Iu(\omega)-D)^{-1}+(Iu(-\omega)-D)^{-1}]P^{-1}M_1\}\underline{1} \quad ,$$

where $u(s) = \exp\{-is\}$, with $i = \sqrt{-1}$.

Substituting from (8.B.2.2),

$$(Iu(\omega)-D)^{-1} + (Iu(-\omega)-D)^{-1}$$

$$= \{\delta_{ij}(u(\omega)-\lambda_j)^{-1} + (u(-\omega)-\lambda_j)^{-1}\delta_{ij}\}$$

$$= \{2\delta_{ij}(\cos\ \omega-\lambda_j)/(1-2\lambda_j\cos\ \omega+\lambda_j^2)\} \qquad (8.B.2.4)$$

It is a simple matter to complete the analysis using a FORMAC program.

Actual computation of D, P and P^{-1} made use of the IBM Scientific Subroutine Package (SSP-360) routines DEIGENP and CMTRIN, the latter modified to perform double precision arithmetic. DEIGENP is a program which computes the eigenvalues and eigenvectors of a general real matrix by means of the Q-R method. (See for example Stewart, 1973). CMTRIN computes the inverse of a complex matrix by the method of Gaussian elimination.

This method worked very successfully with small test matrices, such as that of the semi-Markov generated point process equivalent to the E(5,2) process which had dimension 5 and real eigenvectors and eigenvalues. On larger matrices, such as the semi-Markov generated point process equivalent to the E(5,3) process which has dimension 15, the numerical errors grew to the point that meaningful analysis was impossible. It is possible that some error reduction

techniques might be applied which would render this method useful for larger matrices.

No effort was made to determine if, in general, the stochastic transition matrices considered here are diagonalizable. It was assumed that such a procedure was feasible, presuming that if the assumption were false, it would be indicated by the numerical results.

VIII.B.3. The Interval Spectrum by Numerical Approximation

The definition of the interval spectrum of a semi-Markov generated point process, given by 5.B.12) is

$$f_+(\omega) = \{1+\frac{2}{\sigma^2} \sum_{j=1} \underline{\pi}^T M_1 (T^{j-1}-T^\infty) M_1 \underline{1} \cos j\omega\}/\pi \quad . \quad (8.B.3.1)$$

From (5.B.15), it is known that T^j converges geometrically to T, as j grows large. Let $br^j \geq \max\{|t_{ik}^{(j)}-t_{ik}^{(\infty)}|\}$, for some $0<r<1$. (The existence of b and r are shown by Kemeny and Snell,1960). Then

$$\left| \sum_{j=N+1} \underline{\pi}^T M_1 (T^{j-1}-T^\infty) M_1 \underline{1} \right| \leq \underline{\pi}^T M_1^2 \underline{1} b \sum_{j=N+1}^{\infty} r^j$$

$$\leq \underline{\pi}^T M_1^2 \underline{1} b r^{N-1}/(1-r) \quad . \quad (8.B.3.2)$$

That is, the truncation of the summation in (8.B.3.1) at $j=N$ has error bounded by the right hand side of (8.B.3.2).

The method of numerical approximation outlined here[1] is computationally the most efficient means of generating spectral values. It has the side benefit of providing the first N serial covariances directly.

VIII.C. THE ARBITRARY INTERVAL DENSITY OF THE E(k,p) PROCESS

In Chapter VII, a computational form for the density of an arbitrary interval of an E(k,p) process was given (7.B.6), with the product of two Erlang density functions given by (7.B.5).

Evaluation of (7.B.6) made use of two properties of FORMAC not previously mentioned. First is the capability to evaluate combinatorial expressions such as $\binom{n}{j}$. The second property is the ability to manipulate functions in symbolic form. For example, H(j,n) is defined by

$$H(j,n) = (n\lambda)^j x^{j-1} \exp\{-n\lambda x\}/\{(j-1)!\} \quad . \qquad (8.C.1)$$

Then equation (7.B.6) can be evaluted in terms of H(j,n), performing the operations indicated in (7.B.5) at each step. The right hand side of (8.C.1) can be substituted into the final form for numerical evaluation of the arbitrary interval pdf, or the expression can be examined as a linear combination of Erlang densities, H(j,n). The representation in

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[1]Suggested by Professor W. M. Raike of the Naval Postgraduate School.

which the function H(j,n) was replaced with the Erlang pdf was used to generate the comparative plots shown in Figures VIII.C.1 through VIII.C.4.

Two features of the plots are worth special note. The first is that the initial points are in agreement with the result stated as Theorem VII.A.1. That is, for the arbitrary interval density of an E(k,p) superposition process, with $\lambda=k/p$ the initial point is

$$f_k^{(p)}(0+) = (p-1)/p \quad . \tag{8.C.2}$$

A second point is that regardless of the parameters of the case, each arbitrary interval density is unimodal. While it has not been proven that this holds generally for all k and p, it may be conjectured that this is the case.

It is interesting, and perhaps counter-intuitive, that the distinct shape of the Erlang density functions tends to be retained more with k=2 than with higher values of k. In any event, the special shape is no longer apparent for $k \geq 4$, and the superposition pdf looks much like the exponential.

Because the final form of the density is a convex linear combination of Erlang densities, expectation, variance and coefficient of variation can be obtained directly from the moments of the component distributions without integration. Equation (3.E.8) gave the initial value of the spectrum of counts which can be combined with the expectation and

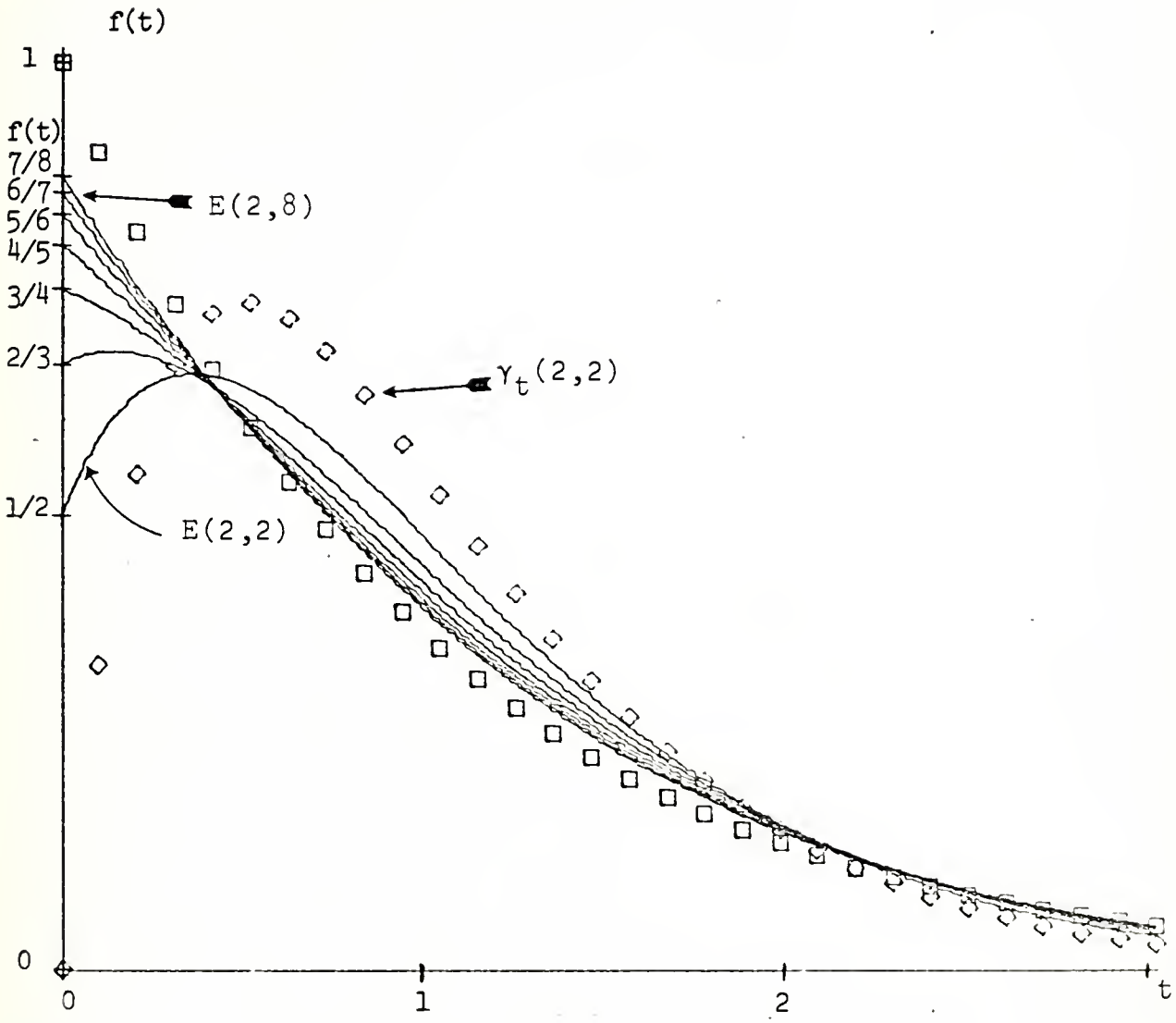

Figure VIII.C.1 Arbitrary interval densities for E(2,p) superposition processes, p=2,...,8. All distributions normalized to unit mean(λ=k/p). The squares represent a unit exponential pdf. The diamonds trace a $\gamma_t(2,2)$ pdf.

The initial points are (p-1)/p for each curve. This figure illustrates the dependence on p as the pdf curves rapidly lose the distinctive Erlang shape and approach the exponential curve.

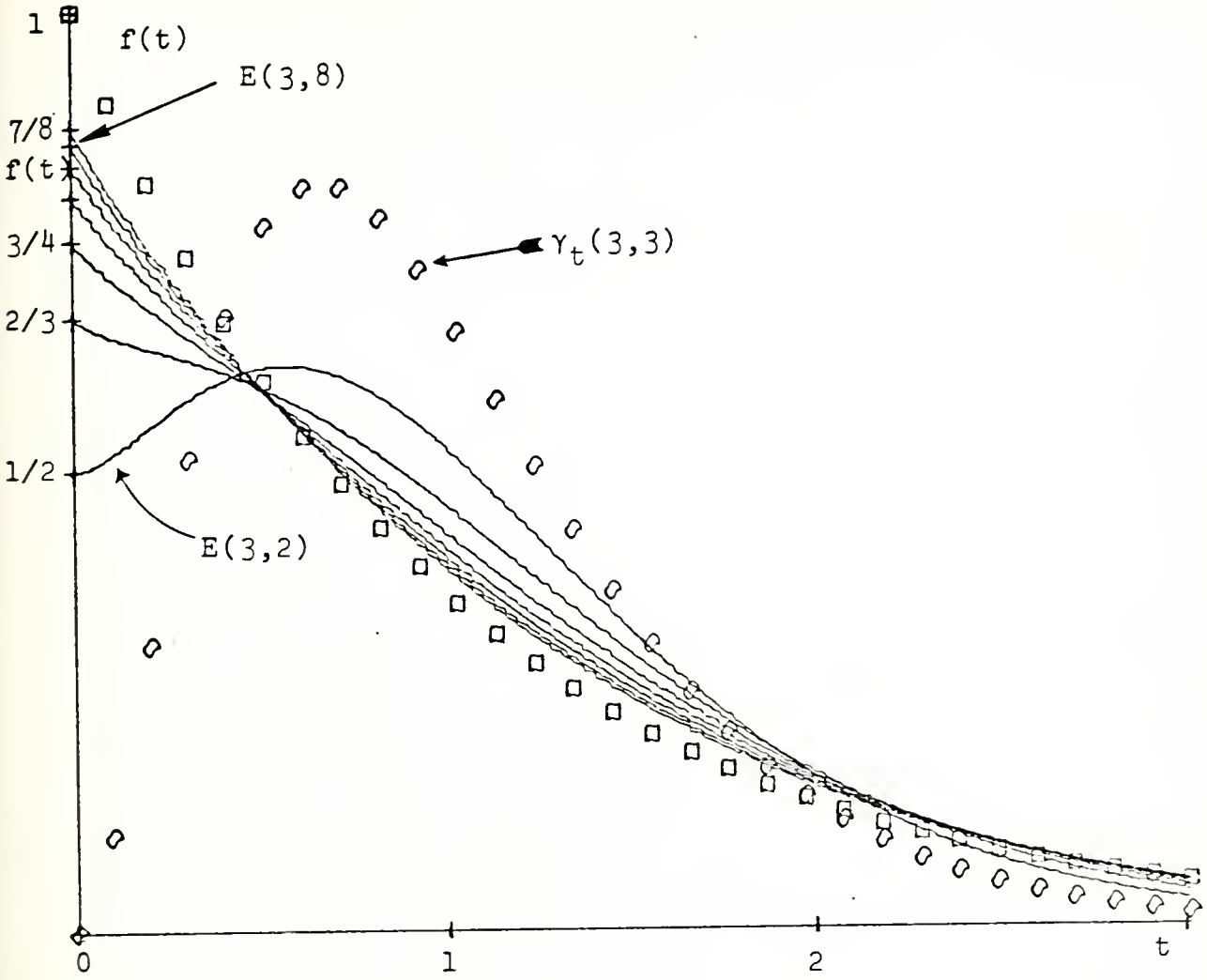

Figure VIII.C.2 Arbitrary interval densities for E(3,p) superposition processes. Squares, diamonds and normalization as in Figure VIII.C.1. With k=3, the distinctive shape of the Erlang pdf fades more rapidly with increased p than for k=2. The trend continues with larger k, so that by k=5, p=3 the pdf looks exponential.

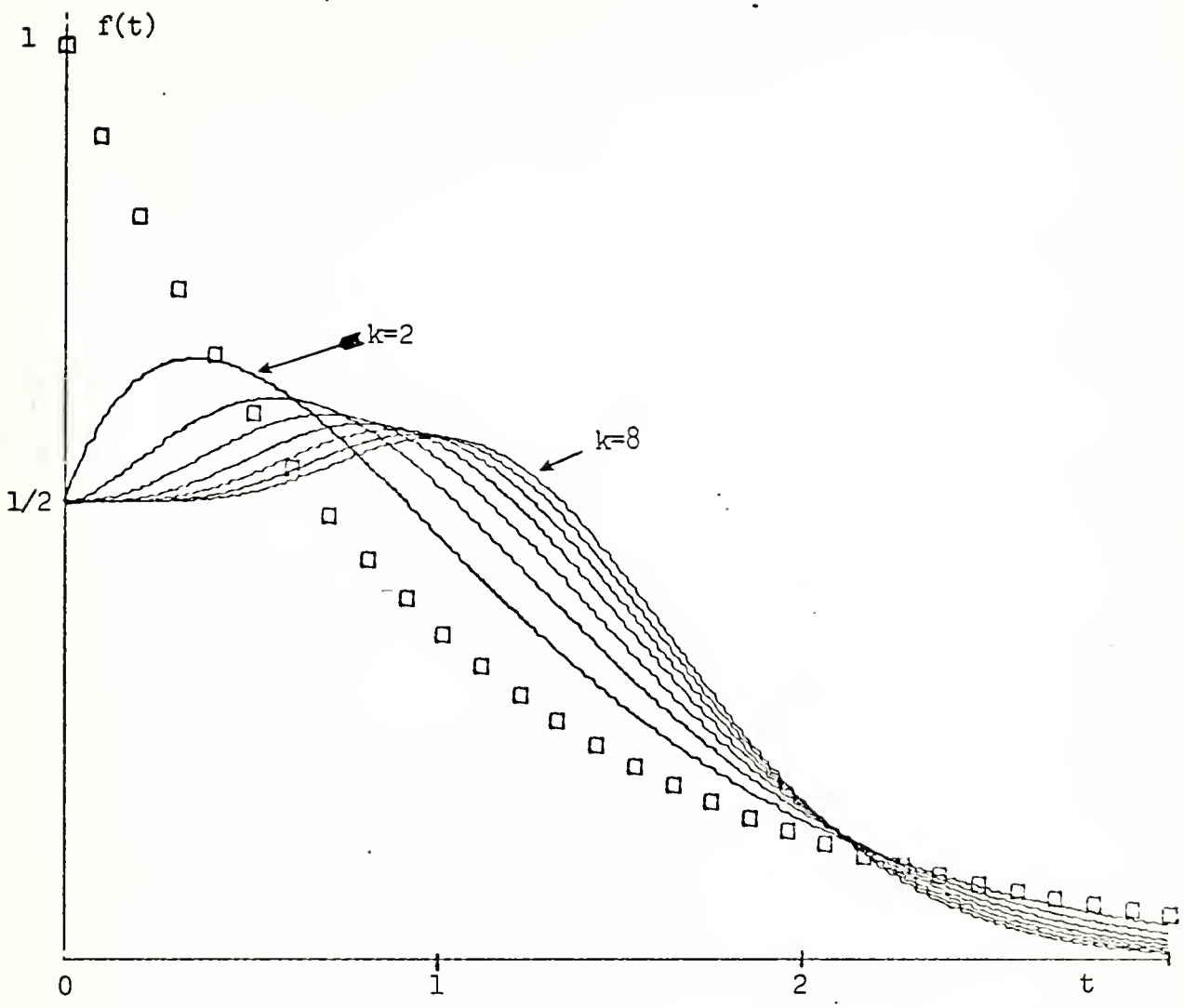

Figure VIII.C.3 Arbitrary interval densities for E(k,2) superposition processes, k=2,...,8. The squares denote a unit exponential pdf. The arbitrary interval pdf's are normalized to $\lambda=k/p$. While each curve has the common initial value of 1/2, the maximum values diminish and are more distant from the origin as the densities seem to be taking on a rectangular form with increasing k.

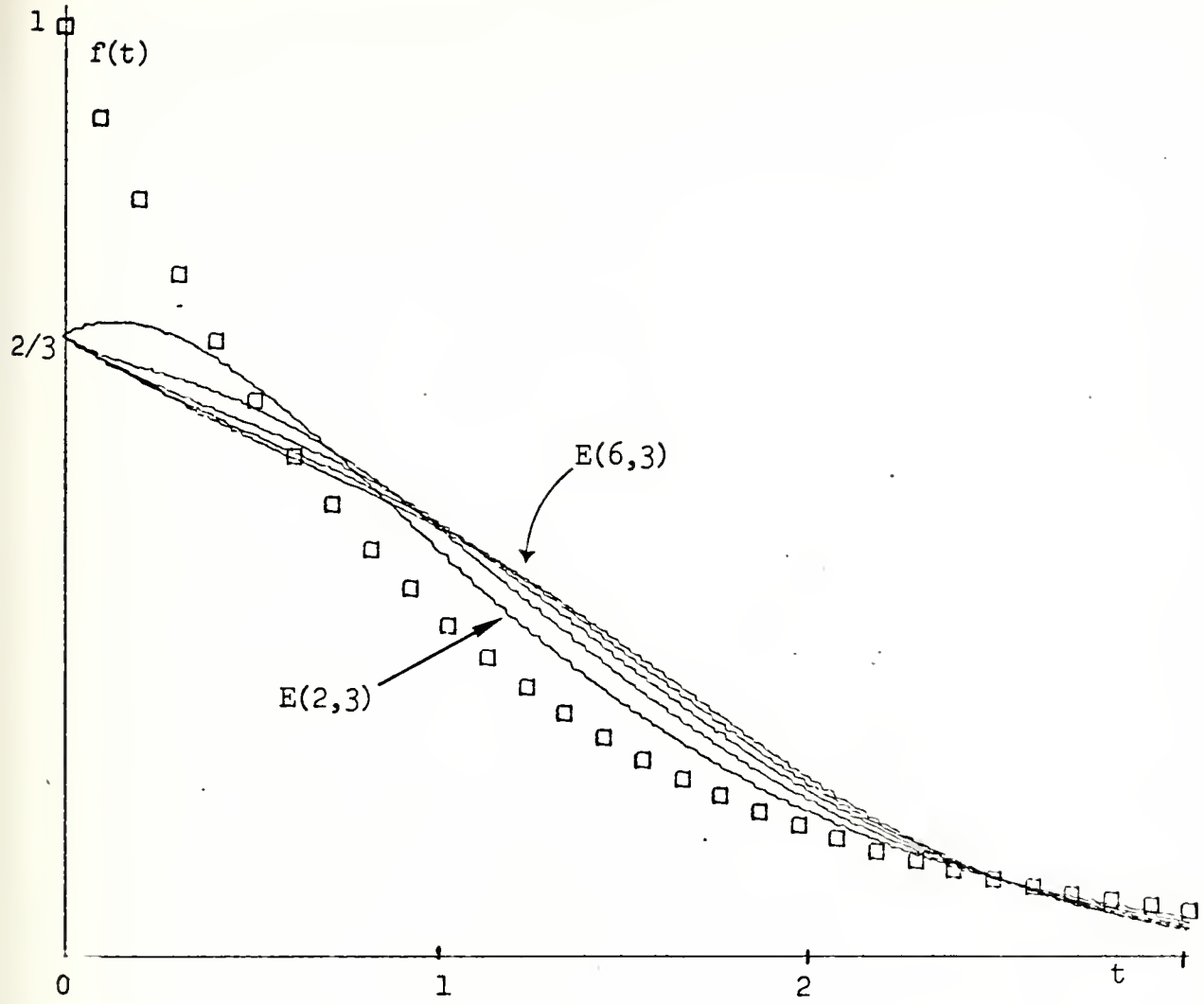

Figure VIII.C.4 Arbitrary interval densities for E(k,3) superposition process, k=2,...,6 normalization as in Fig. VIII.C.3. Here the node associated with the Erlang pdf has virtually disappeared for k>2. When p is increased beyond 3, the pdf's of the various processes are nearly indistinguishable, differing only slightly from the exponential.

k	p	E[x]	V[x]	$C^2[x]$	$g_+(0+)$	$f_+(0+)$
2	2	$1/\lambda$	$.625/\lambda^2$	.625	$.159\lambda$	.255
2	3	$2/3\lambda$	$.309/\lambda^2$	.694	$.239\lambda$	.229
2	4	$1/2\lambda$	$.185/\lambda^2$	.740	$.318\lambda$	.215
2	5	$2/5\lambda$	$.124/\lambda^2$	.773	$.398\lambda$	.206
2	6	$1/3\lambda$	$.088/\lambda^2$	.798	$.477\lambda$	.199
4	2	$2/\lambda$	$1.82/\lambda^2$	.455	$.0398\lambda$	.175
4	3	$4/3\lambda$	$1.01/\lambda^2$	.568	$.0596\lambda$	.140
4	4	$1/\lambda$	$.643/\lambda^2$	.643	$.0796\lambda$	.124
4	5	$4/5\lambda$	$.445/\lambda^2$	.694	$.0995\lambda$	.115
4	6	$2/3\lambda$	$.326/\lambda^2$	.734	$.1194\lambda$	.108
6	2	$3/\lambda$	$3.64/\lambda^2$	.405	$.0177\lambda$	.131
6	3	$2/\lambda$	$2.14/\lambda^2$	.536	$.0265\lambda$	.099
6	4	$3/2\lambda$	$1.39/\lambda^2$	.620	$.0354\lambda$	.095

Figure VIII.C.5 Data derived from E(k,p) pdf analyses. Observe that $C^2(x)$ decreases with k and increases with p, while $f_+(0+)$ decreases with increase in either k or p.

coefficient of variation as in (7.B.11) to provide the value of the initial point of the interval spectrum.

The quantities thus obtained were used to verify those obtained through spectral analysis. In each case, positive verification was achieved.

A summary of the key values related to pdf analysis for certain E(k,p) processes are tabulated in Figure VIII.C.5.

VIII.D. NUMERICAL REPRESENTATION OF THE INTERVAL BISPECTRUM OF A SEMI-MARKOV GENERATED POINT PROCESS

In Chapter V, Section C, the procedure for generating higher order spectra was outlined, with a specific form given for the interval bispectrum of a semi-Markov generated point process (5.C.10). Evaluation of (5.C.10) hinges, as with the interval spectrum, on the matrix $(Iu-T)^{-1}$, as in Section B of this chapter. For this reason, any of the treatments of Section VIII.B may be applied to this problem. For purposes of illustration, the numerical approximation method (VIII.B.3) was selected to evaluate and plot the bispectrum of intervals in the E(5,2), E(5,3) and E(3,5) processes from their semi-Markov generated point process representation.

The results of this procedure are illustrated in Figures VIII.D.1 through VIII.D.6. It will be recalled from Section V.C that the bispectrum may be defined on any one of twelve sectors of the real plane. The examples illustrated in this section are shown in the sector defined by $0 \leq \omega_2 \leq \omega_1 \leq \pi$.

Two views are shown of each bispectru, the first is the view from the positive quadrant back toward the origin. The second view is from the negative quadrant looking over the positive quadrant.

Since the bispectrum is a complex valued function, it was decided to plot the absolute value of each point. Also, the bispectral values were scaled by a factor of 10 to exagerate the surface shape. With the scale for all axes the same, the absolute bispectrum looks almost flat, a matter of some concern if the bispectrum of some data is to be estimated.

VIII.E. INTERVAL SERIAL CORRELATION COEFFICIENTS

In a mixed autoregressive/moving average process of orders m and n respectively, it was noted in Section III.E that the serial correlation coefficients for lags greater than n can be expressed as

$$\rho_k = a_1\rho_{k-1} + \ldots + a_m\rho_{k-m} \quad , \quad k>n \qquad (8.E.1)$$

A major portion of model identification analysis involves determining n, m and $\{a_j\}$ from the sequence of serial correlation estimates (Box and Jenkins(1970)). Where n and m and $\{\rho_j\}$ are known, the coefficients a_j, can be determined from the linear system

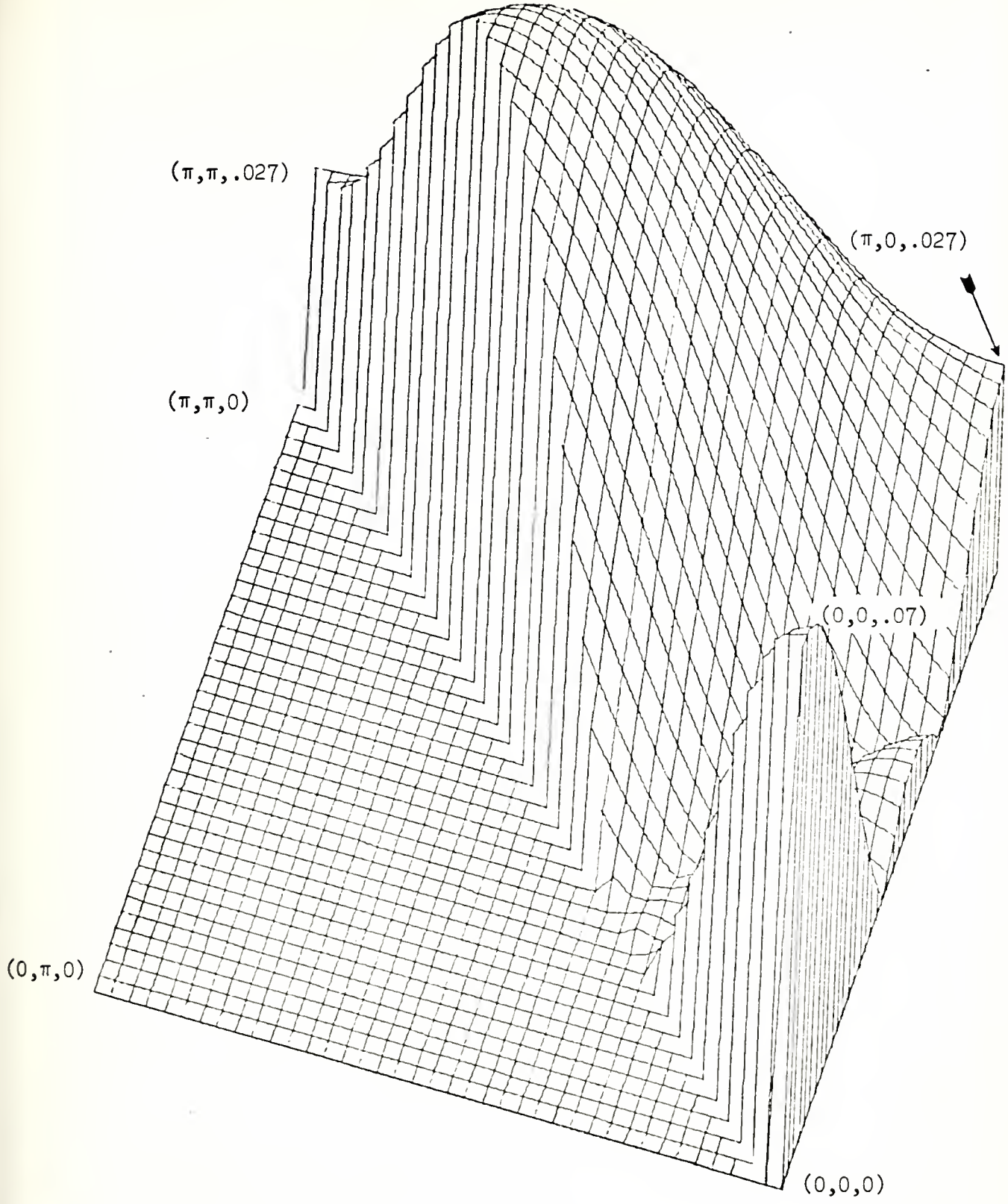

Figure VIII.D.1
Absolute bispectrum of the E(5,2) interval process. This and Fig. VIII.D.2 are isometric projections of the E(5,2) bispectrum. Only one octant is shown as the values are reflected in the adjacent octant. Interpretation of this type of representation remains an open question. Construction of this and the following bispectral representations was by method of approximation.

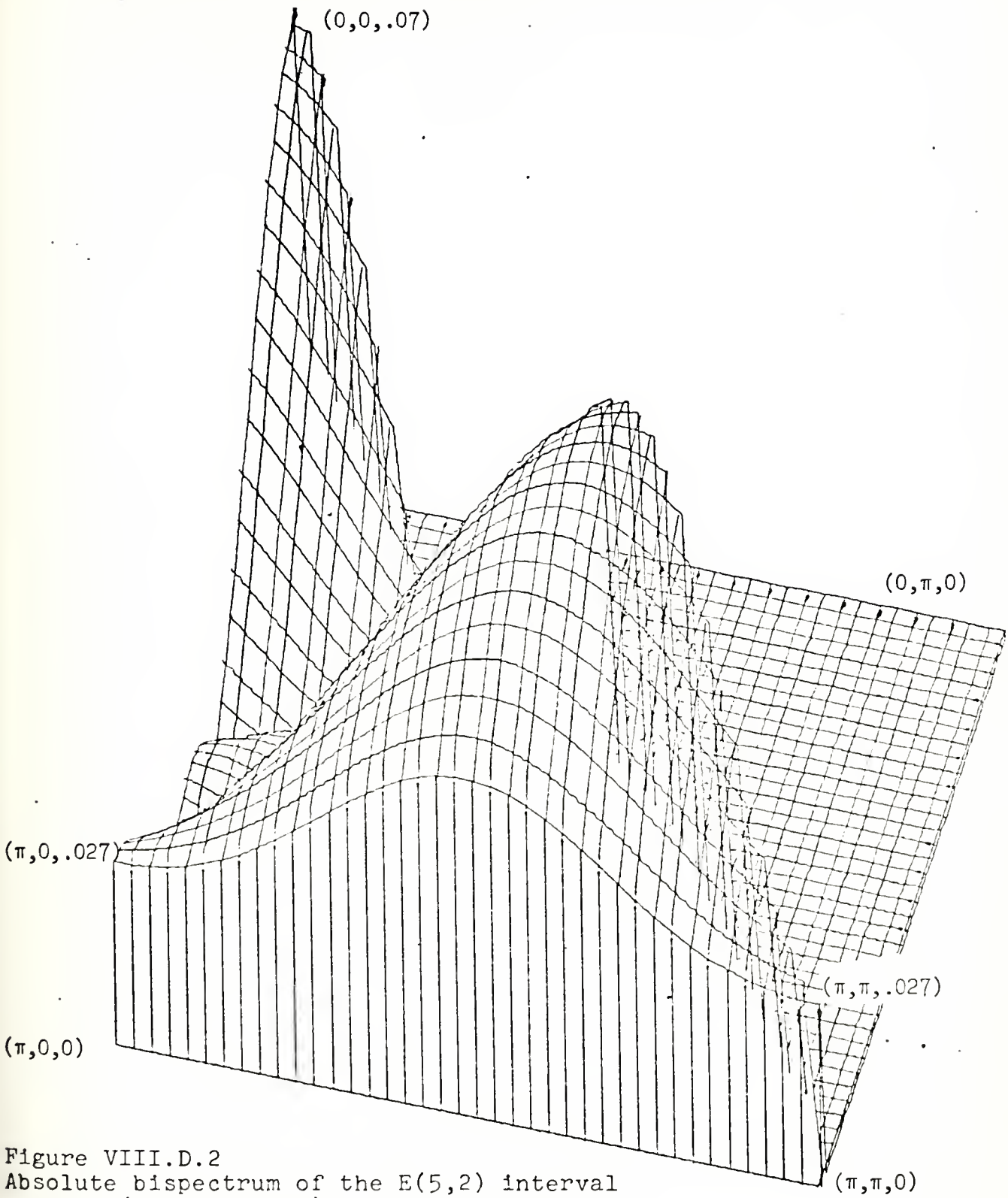

Figure VIII.D.2
Absolute bispectrum of the E(5,2) interval process (Reverse view). This is a view of the E(5,2) absolute bispectrum of intervals looking toward the origin from the first quadrant.

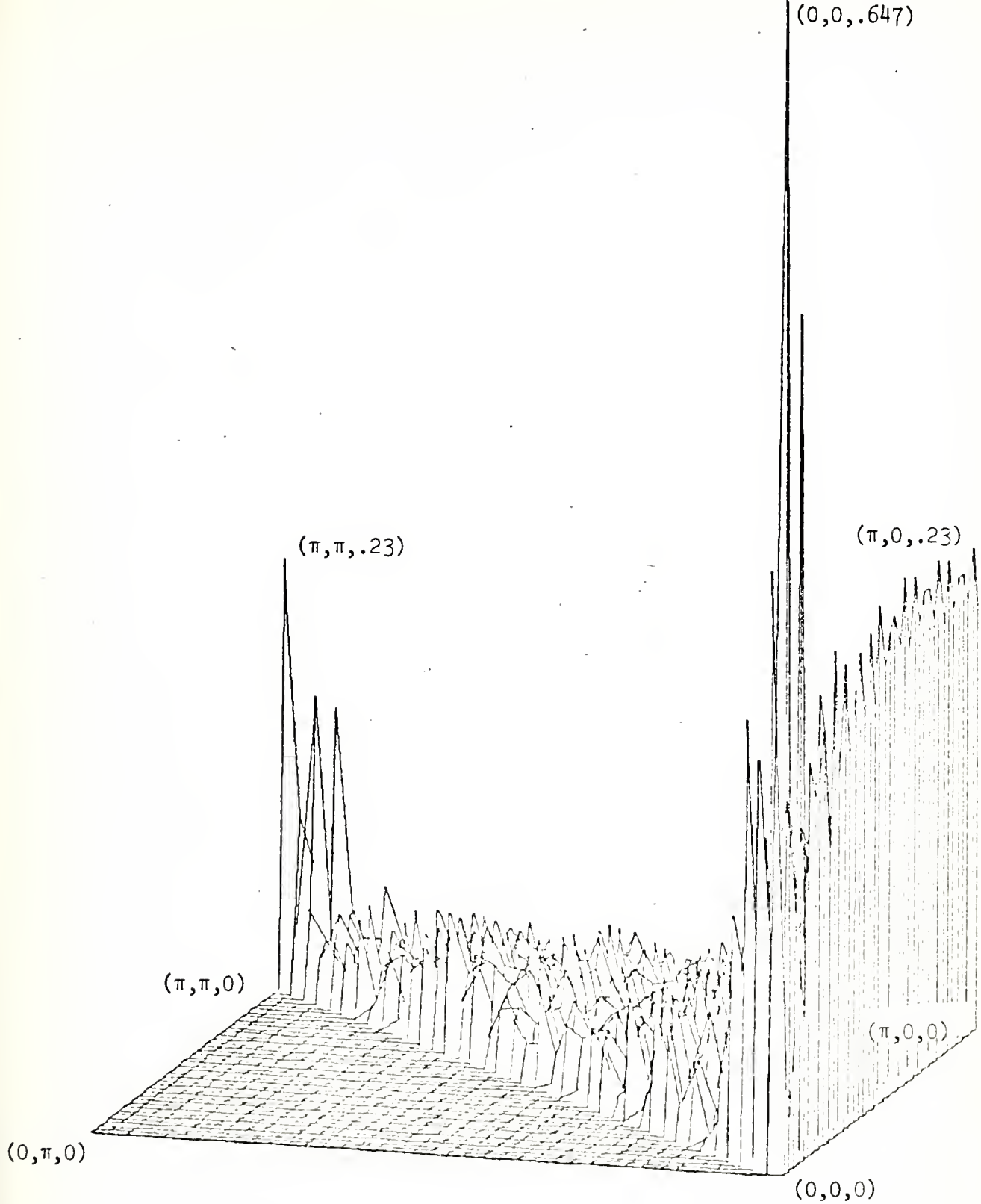

Figure VIII.D.3 Absolute bispectrum of intervals of the E(5,3) process. This and Figure VIII.D.4 represent the absolute bispectrum of the E(5,3) process. The jagged appearance may be due in part to the course grid used in the plot (40 x 40).

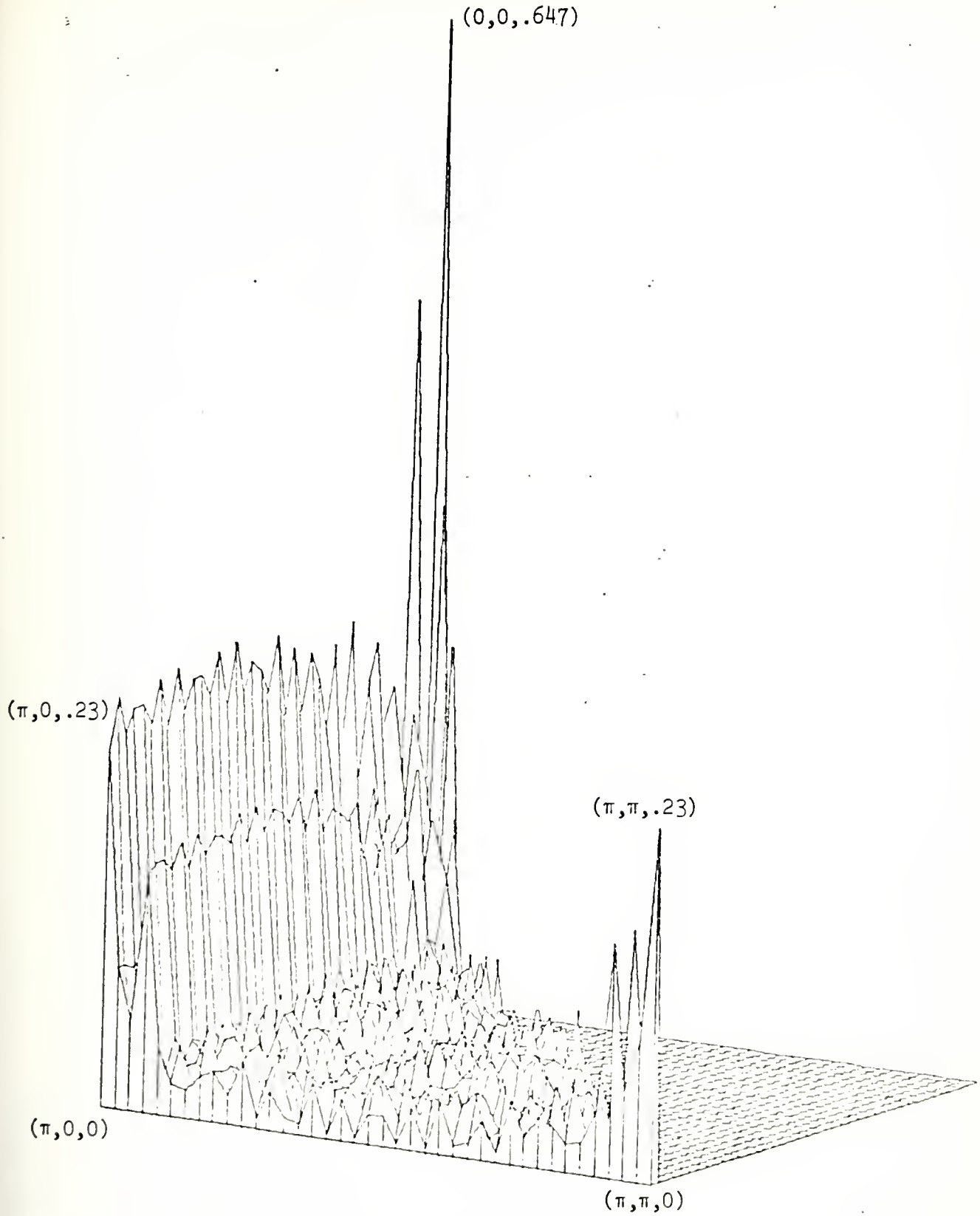

Figure VIII.D.4 Absolute bispectrum of intervals for the E(5,3) process (reverse view). This is a view of the E(5,3) absolute bispectrum looking back toward the origin from the first quadrant. Note that the magnitude of the peaks is about 10 times that of the E(5,2) process.

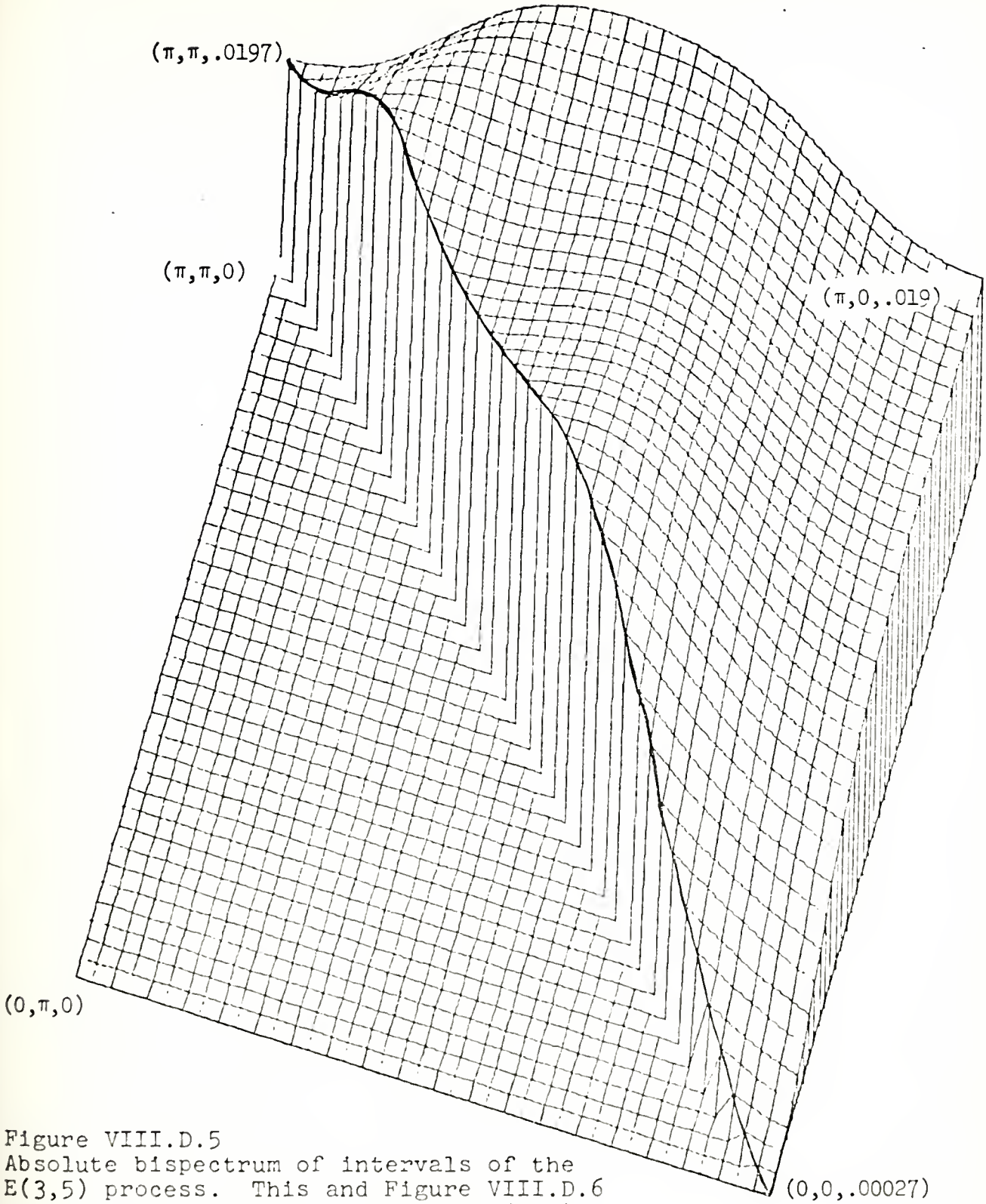

Figure VIII.D.5
Absolute bispectrum of intervals of the E(3,5) process. This and Figure VIII.D.6 are isometric projections of the E(3,5) absolute interval bispectrum. Note the relatively flat appearance, nearly vanishing at the origin. This is similar to the second order spectrum which flattens with increased p.

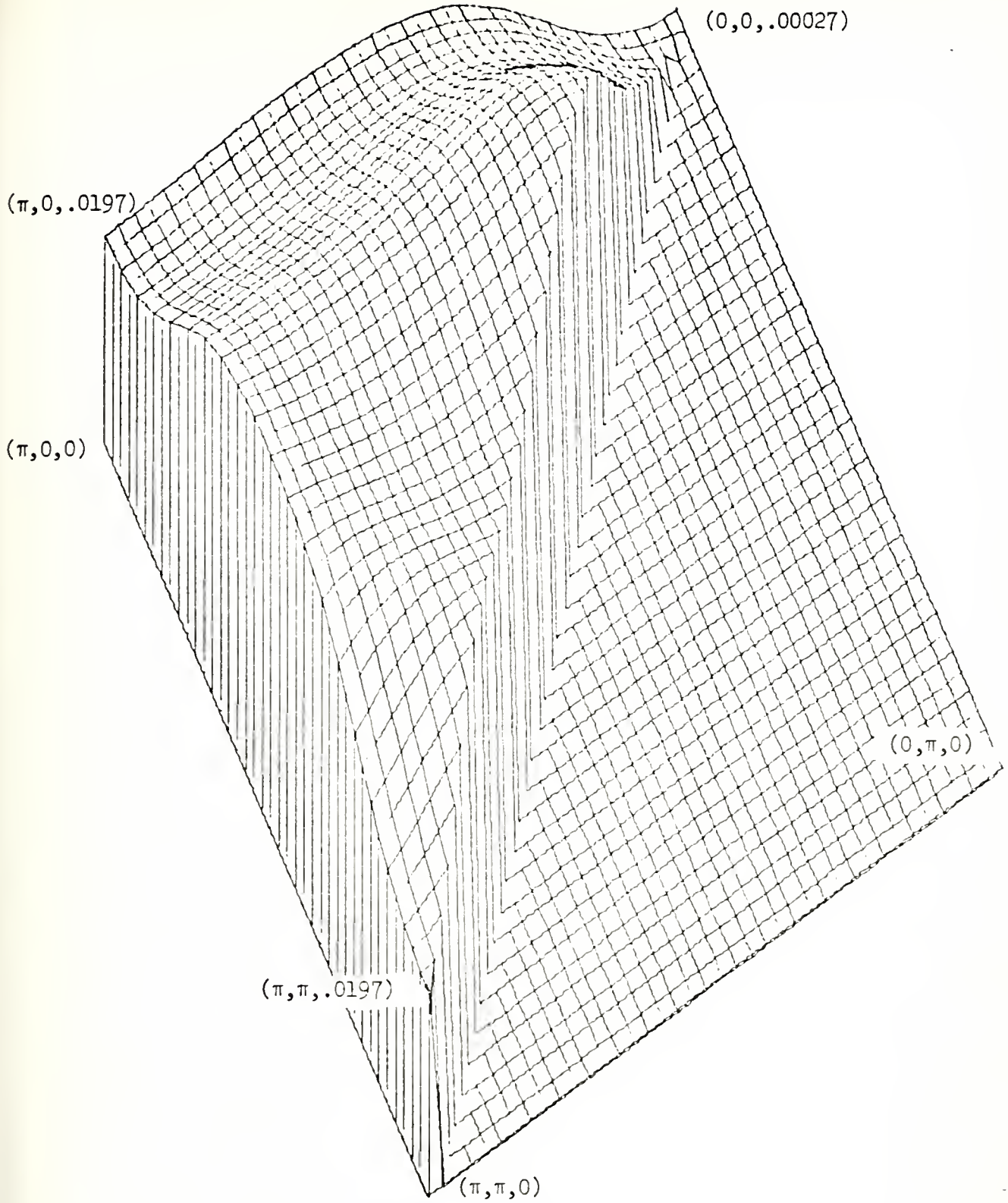

Figure VIII.D.6 Absolute bispectrum of the E(3,5) interval process (reverse view). Observe that the surface is slightly wavey with the highest peak at $(\pi/2,\pi/2)$, with secondary peaks at $(3\pi/4,3\pi/4)$, $(\pi,\pi/2)$ and $(\pi,3\pi/4)$.

$$\begin{vmatrix} \rho_n & \cdots & \rho_{n+m-1} \\ \vdots & & \vdots \\ \rho_{n-m+1} & \cdots & \rho_n \end{vmatrix} \times \begin{vmatrix} a_1 \\ \vdots \\ a_m \end{vmatrix} = \begin{vmatrix} \rho_{n+1} \\ \vdots \\ \rho_{n+m} \end{vmatrix} \qquad (8.E.2)$$

If the actual order of the process is (m',n'), the following result is obtained from solution of the system given by (8.E.2):

$n' \leq n$ and $m > m'$:	The matrix of correlation coefficients will be singular
$n' > n$ or $m' > m$:	Equation (8.E.1) will not be satisfied for $k > n+m$ using the coefficients derived from (8.E.2).

This idea leads to a procedure for verification of the results of Section III.D wherein a method is presented for determining the order of the polynomials which make up the interval spectrum of an E(k,p) process.

When the process being studied has been given an equivalent semi-Markov generated point process representation, the serial coefficients may be obtained directly form (5.B.3) and (5.B.4) which give the covariance and variance as functions of powers of the transition matrix, T. If only a spectral representation is available, as for example output

of SPECD (Subsection VIII.A.1), the spectrum must be inverted to produce the serial correlation coefficients.

Recall from (2.A.3.3) that

$$f_+(\omega) = \{1 + 2 \sum_{j=1}^{\infty} \rho_j \cos j\omega\} / \pi \quad , \quad 0<\omega\leq\pi \ . \qquad (8.E.3)$$

The inverse relation (see for example Cox and Lewis(1966, Ch. 4)) is

$$\rho_j = \int_0^{\pi} f_+(\omega) \cos j\omega \, d\omega \qquad (8.E.4)$$

A discrete approximation of (8.E.3)

$$f_+(\omega_n) = \{1 + 2 \sum_{j=1}^{N} \pi_j \cos j\omega_n\}/\pi \quad , \qquad (8.E.5)$$

where $\omega_n = \pi n/N$, $n=0,\ldots,N-1$. The inverse relation of (8.E.5) is

$$\rho_j = \frac{\pi}{N} \sum_{n=0}^{N-1} f_+(\omega_n) \cos j\omega_n \ , \quad j=0,1,\ldots,N-1 \ . \qquad (8.E.6)$$

where π/N represents the separation between ω_n and ω_{n+1}.

The equations (8.E.5) and (8.E.6) represent a finite Fourier transform pair, with communication between the two representations easily handled by the Fast Fourier Transform Algorithm (See for example Cooley, Lewis and Welch (1967)).

The key property of the finite Fourier transform which is the basis for the fast Fourier transform is that the transform of a sequence of $N=2^n$ points can be written as the sum of n transforms of lengths 2, 2, 2^2, 2^3,..., 2^{n-1} . Since each transform of length r requires approximately r^2 operations, the number of steps required by the fast Fourier transform is $nN=N\log_2 N$ as compared with an N point transform requiring N^2 operations.

The program package FORT/RFORT supplied by IBM was used in this analysis. Figure VIII.E.1 is a comparison of the E(8,2) process correlation structure as determined by SPECD, FORT/RFORT, and as determined by the approximation method applied to the semi-Markov representation of this process. The parameter k=8 accentuates the differences in the procedures which are indistinguishable for smaller k.

The correlation structure of additional processes is tabulated in Appendix D.

VIII.F. COUNT SPECTRA OF THE E(k,p) PROCESS

The second order count spectrum of an E(k,p) process is given by (3.C.6). Direct numerical evaluation of this form for various values of k is shown in Figure VIII.F.1. Because the count spectrum is linear in p, a value of p=1 has been used for all computations. Two features are readily apparent from inspection of Figure VIII.F.1. The first is that for small k after the initial fluctuation the spectrum of counts converges rapidly to $1/\pi$. This fact is readily verified by taking the limit $\omega\to\infty$ in Equation (3.C.7).

Figure VIII.E.1 Comparison of SPECD and the approximation method of spectral representation.

The accuracy and efficiency of two methods used for numerical approximation of the interval second order spectrum of the E(k,p) process are compared for the E(8,2) process.

Method 1. Numerical approximation of the spectrum using SPECD (VIII.A.1). Fast Fourier transform of spectrum to determine correlations and damping coefficients (32,768 and 4096 point approximations)

Method 2. Numerical approximation of the equivalent semi-Markov generated point process (VIII.B.3)

		METHOD 1		METHOD 2
Number of points		32,768	4096	200
CPU Time		8 min	1 min	6 sec
Core		448K	202K	248K
Spectrum		Same	Same	Same
Correlations:	1	-0.52122	-0.52123	-0.52123265925713
	2	0.28153	0.28154	0.28153949321523
	3	-0.14812	-0.14812	-0.14813039868207
	4	0.078694	0.078695	0.078699294631380
Coefficients	1	-0.0091941	-0.011426	0.000015258401560525
	2	0.13558	0.16640	0.032958990549093
	3	-0.24262	-0.17664	-0.46874998699262
Error		10^{-6}	10^{-5}	10^{-14}

The differences in precision between Methods 1 and 2 lie in the fact that only single precision results are returned from SPECD and FORT/RFORT although double precision arithmetic is used in both programs.

The E(8,2) case accentuates differences. In those cases for which the spectrum has a flatter appearance, the damping coefficients are nearly the same from either method.

The error is based on the comparisons

$$\rho_{n+1} : \bar{\rho}_{n+1} = \rho_n \alpha_1 + \ldots + \rho_{n-m+1} \alpha_n$$

and

$$\rho_{n+2} : \bar{\rho}_{n+2} = \rho_{n+1} \alpha_1 + \ldots + \rho_{n-m+2} \alpha_m$$

The second observation from Figure VIII.F.1 is that the count spectra doe not exhibit the strong dependence on k observable in the interval spectra.

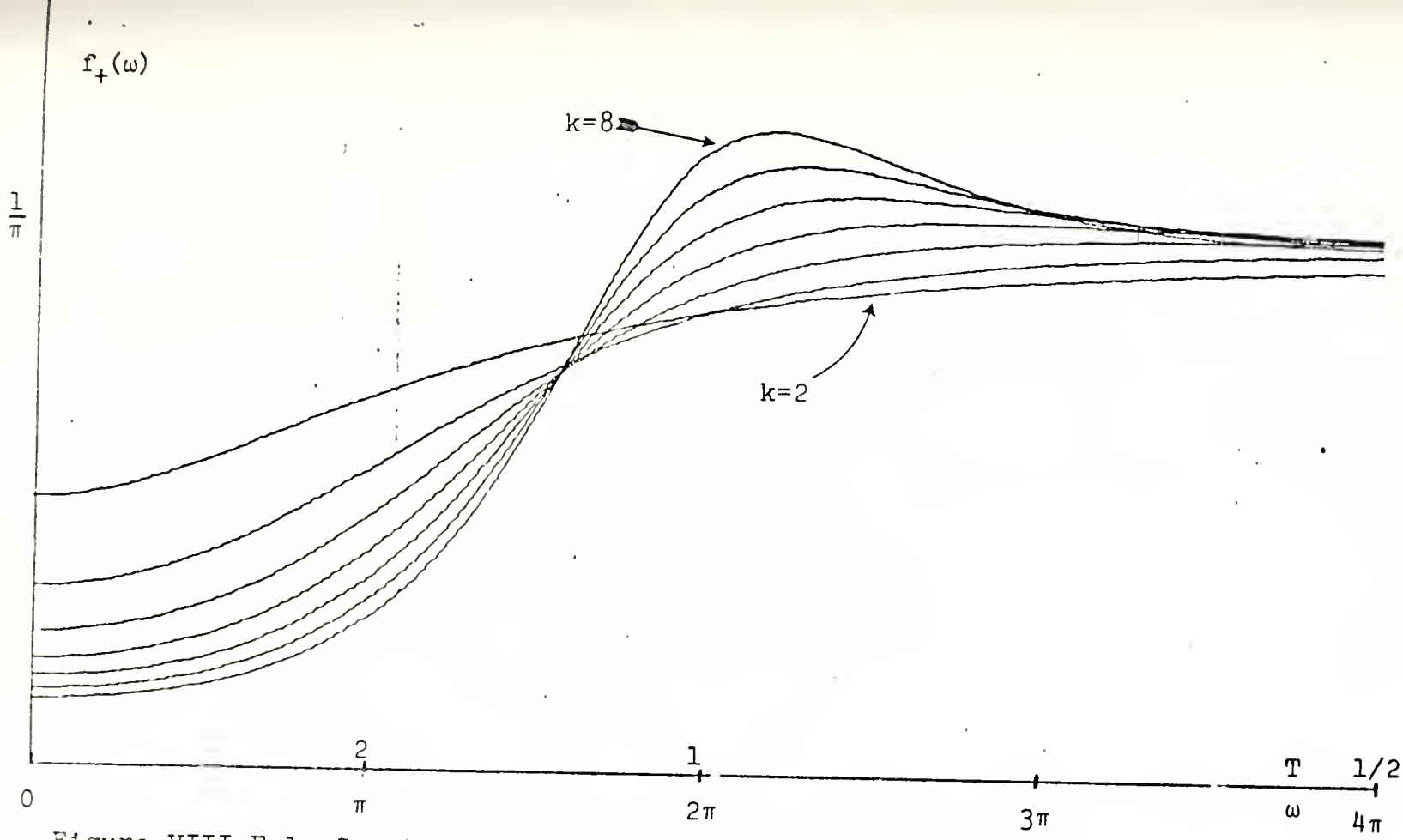

Figure VIII.F.1 Count spectra of Erlang renewal processes, k=2,...,8. The count spectra of Erlang renewal processes with λ=k is shown for small k. As k increases, the process becomes pseudo periodic with a peak developing at T=1 ($\omega=2\pi$). For large values of k (k>25) harmonics will appear at 4π, 6π, etc.

APPENDIX A NOTATION

1. General Forms

$F(x)$, $F_X(x)$: probability distribution functions

$R(x)$, $R_X(x)$: survivor functions

$Z(x)$: equilibrium excess (forward recurrence time) distribution

$f(x)$, $f_X(x)$: probability density functions

$E(\cdot)$, $Var(\cdot)$: expected value and variance

$o(h)$: a function that vanishes more rapidly than h

$C(\cdot)$: coefficient of variation

2. Conventions

Superscript *: Laplace transform

Superscript (p): The superscripted symbol refers to a superposition point process

λ, λ_j: scale parameters of Erlang and hyper-exponential distributions

μ: the expected value of an interval in a renewal process, or an arbitrary interval in a stationary point process

σ^2: the variance of an interval in a renewal process, or an arbitrary interval in a stationary point process

$\lfloor x \rfloor$: the largest integer less than or equal to x

iid: independent, identically distributed; refers to either random variables or component processes in a superposition

Script upper case letters: sets; vectors

Script lower case letters: sets, vectors

Underlined lower case letters: vectors

Gothic, upper case letters: matrices

Reserved symbols: Unless otherwise specifically noted, k is the shape parameter of an Erlang distribution; p is the number of components in a superposition

3. Point Process Notation

$N(t)$: asynchronous counting process (II.A.1)

$N_f(t)$: synchronous counting process (II.A.1)

$\{X_n\}$: synchronous interval process (II.A.1)

$\{W, L_1, L_2, \ldots\}$: asynchronous interval process (II.A.1)

$M(t)$: asynchronous mean-time curve (2.A.2.2)

$m(t), m$: asynchronous intensity (2.A.2.5)

$M_f(t)$: synchronous mean-time curve (2.A.2.4)

$m_f(t)$: synchronous intensity (2.A.2.6)

$V(t)$: variance-time curve (2.A.2.3)

$\gamma(\tau)$: covariance density of the counting process (2.A.2.10)

$g_+(\omega)$: second order spectrum of counts (2.A.2.13)

$\phi(z,t)$: generating function of the asynchronous counting process (2.A.3.4)

$f_t(\omega)$: second order spectrum of counts (2.A.3.3)

$\{\rho_j\}$: serial correlation sequence (2.A.3.7)

4. Semi-Markov Generated Point Process Notation

$T = \{t_{ij}\}$: Transition matrix (V.A)

$F(u) = \{f_{ij}(u)\}$: transition time pdf matrix (V.A)

$E_r = \{\mu_{ij}^{(r)}\} = \{E\, S_{ij}^{r}\}$: matrix of r-th transition time moments (V.A)

$\underline{\pi}$: stationary vector associated with (5.A.1.4)

$G(u) = \{t_{ij}f_{ij}(u)\}$: weighted transition time pdf matrix (V.B)

$M_r = \{t_{ij}\mu_{ij}^{(r)}\}$: weighted r-th moment matrix (V.B)

T^{∞}: the steady state transition matrix (5.B.2)

$\gamma(j)$: interval covariance of j lags (5.B.9)

$\tau(j,k)$: interval trivariance of j and k lags (5.C.1)

$\tau^*(a,b)$: Laplace transform of $\tau(j,k)$, $0 < j < k$ (5.C.15)

$\tau^*(a)$: Laplace transform of $\tau(j,j)$, $0 < j$ (5.C.10)

$\tau_o^*(a)$: Laplace transform of $\tau(o,j)$, $0 < j$ (5.C.17)

a:b: transition from state a to b

$b(\omega_1,\omega_2)$: the bispectrum of intervals

5. Special Notation

$E(k,p)$: the superposition of p iid k-Erlang renewal processes (II.A.5)

$H(k,p;\underline{q},\underline{\lambda})$: the superposition of p iid hyperexponential processes with parameter vector $\underline{\lambda}$ and probability mass function $\underline{q}$. (II.A.5)

ARMA(m,n): an autoregressive/moving average process of degrees m and n respectively (I.F.1)

a_J, b_J, c_J: individual terms in the E(k,p) interval spectrum (3.A.1.23)

a_L, b_L, c_L: individual terms in the E(k,p) interval spectrum: revised index (3.A.2.3)

$\nu^{k/2}(p)$: special index set for the E(k,p) interval spectrum - k even (3.C.3)

$\gamma_t(k,\lambda)$: the k-Erlang pdf (2.A.5.1)

APPENDIX B: COSINE CONVERSION FORMULA

1. Convert $\cos^k\omega$ to $\sum_{j=0}^{k} \alpha_{kj}\cos j\omega$

$$\cos^k\omega = \begin{cases} 2^{-k}(\alpha_{k0}+\alpha_{k2}\cos 2\omega+\alpha_{k4}\cos 4\omega+\ldots+\alpha_{kk}\cos k\omega), & k \text{ even} \\ 2^{-k}(\alpha_{k1}\cos \omega+\alpha_{k3}\cos 3\omega+\ldots+\alpha_{kk}\cos k\omega), & k \text{ odd} \end{cases}$$

$$\alpha_{k0} = \alpha_{k-1,1} \quad , \; k=2,4,6,\ldots$$

$$\alpha_{k1} = 2\alpha_{k-1,0} + \alpha_{k-1,2} \quad , \; k=3,5,\ldots$$

$$\alpha_{kn} = \alpha_{k-1,n-1} + \alpha_{k-1,n+1} \quad , \; n=2,3,\ldots,k-1 \; ; \quad k=3,4,5,\ldots$$

$$\alpha_{nk} = 1 \quad , \quad k=1,2,3,4,\ldots$$

$$\alpha_{10} = \alpha_{01} = 1$$

2. Convert $\cos k\omega$ to $\sum_{j=0}^{k} \alpha_j\cos^j\omega$ (Tschebysheff Polynomials)

$$\cos 0\omega = 1$$

$$\cos 1\omega = \cos \omega$$

$$\cos k\omega = 2\cos \omega \cos\{(k-1)\omega\} - \cos\{(k-2)\omega\}, \quad k=3,4,5$$

APPENDIX C

Transition probability matrix of the semi-Markov generated point process equivalent to the E(2,3) process

$$\begin{pmatrix} \frac{1}{3} & \frac{4}{3^2} & \frac{2}{3^2} \\ \frac{1}{3} & \frac{4}{3^2} & \frac{2}{3^2} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(2,3) process

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(2,4) process

$$\begin{pmatrix} \frac{1}{4} & \frac{6}{4^2} & \frac{18}{4^3} & \frac{6}{4^3} \\ \frac{1}{4} & \frac{6}{4^2} & \frac{18}{4^3} & \frac{6}{4^3} \\ 0 & \frac{2}{4} & \frac{6}{4^2} & \frac{2}{4^2} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(2,4) process

$$\begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(2,5) process

$$\begin{pmatrix} \frac{1}{5} & \frac{8}{5^2} & \frac{36}{5^3} & \frac{96}{5^4} & \frac{24}{5^4} \\ \frac{1}{5} & \frac{8}{5^2} & \frac{36}{5^3} & \frac{96}{5^4} & \frac{24}{5^4} \\ 0 & \frac{2}{5} & \frac{9}{5^2} & \frac{24}{5^3} & \frac{6}{5^3} \\ 0 & 0 & \frac{3}{5} & \frac{8}{5^2} & \frac{2}{5^2} \\ 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(2,5) process

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(3,2) process

$$\begin{pmatrix} \frac{1}{2^2} & \frac{3}{2^3} & \frac{3}{2^3} \\ \frac{1}{2^2} & \frac{3}{2^3} & \frac{3}{2^3} \\ \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^2} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(3,2) process

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(3,3) process

$$\begin{pmatrix} \frac{1}{3^2} & \frac{2}{3^2} & \frac{4}{3^3} & \frac{4}{3^3} & \frac{20}{3^4} & \frac{10}{3^4} \\ \frac{1}{3^2} & \frac{2}{3^2} & \frac{4}{3^3} & \frac{4}{3^3} & \frac{20}{3^4} & \frac{10}{3^4} \\ \frac{1}{3} & \frac{2}{3^2} & \frac{4}{3^3} & \frac{2}{3^3} & \frac{4}{3^3} & \frac{2}{3^3} \\ 0 & \frac{2}{3^2} & \frac{4}{3^3} & \frac{5}{3^3} & \frac{8}{3^3} & \frac{4}{3^3} \\ 0 & \frac{1}{3} & \frac{2}{3^2} & \frac{1}{3^2} & \frac{2}{3^2} & \frac{1}{3^2} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{2}{3^2} & \frac{1}{3^2} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(3,3) process

$$\begin{pmatrix} 3 & 4 & 5 & 5 & 6 & 7 \\ 2 & 3 & 4 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(3,5) process

$$
\begin{pmatrix}
\frac{1}{5^2} & \frac{12}{5^3} & \frac{24}{5^4} & \frac{72}{5^4} & \frac{72}{5^4} & \frac{108}{5^5} & \frac{48}{5^4} & \frac{432}{5^5} & \frac{1512}{5^6} & \frac{2016}{5^7} & \frac{72}{5^5} & \frac{1008}{5^6} & \frac{6048}{5^7} & \frac{18144}{5^8} & \frac{4536}{5^8} \\
\frac{1}{5^2} & \frac{12}{5^3} & \frac{24}{5^4} & \frac{72}{5^4} & \frac{72}{5^4} & \frac{108}{5^5} & \frac{48}{5^4} & \frac{432}{5^5} & \frac{1512}{5^6} & \frac{2016}{5^7} & \frac{72}{5^5} & \frac{1008}{5^6} & \frac{6048}{5^7} & \frac{18144}{5^8} & \frac{4536}{5^8} \\
\frac{1}{5} & \frac{4}{5^2} & \frac{8}{5^3} & \frac{12}{5^3} & \frac{72}{5^4} & \frac{108}{5^5} & \frac{24}{5^4} & \frac{288}{5^5} & \frac{216}{5^5} & \frac{288}{5^6} & \frac{24}{5^5} & \frac{96}{5^5} & \frac{648}{5^6} & \frac{2016}{5^7} & \frac{504}{5^7} \\
0 & \frac{2}{5^2} & \frac{4}{5^3} & \frac{3}{5^2} & \frac{72}{5^4} & \frac{108}{5^5} & \frac{54}{5^4} & \frac{468}{5^5} & \frac{324}{5^5} & \frac{432}{5^6} & \frac{84}{5^5} & \frac{228}{5^5} & \frac{54}{5^4} & \frac{4032}{5^7} & \frac{1008}{5^7} \\
0 & \frac{1}{5} & \frac{2}{5^2} & \frac{3}{5^2} & \frac{18}{5^3} & \frac{27}{5^4} & \frac{6}{5^3} & \frac{72}{5^4} & \frac{54}{5^4} & \frac{72}{5^5} & \frac{6}{5^4} & \frac{24}{5^4} & \frac{162}{5^5} & \frac{504}{5^6} & \frac{126}{5^6} \\
0 & 0 & \frac{2}{5} & 0 & \frac{6}{5^2} & \frac{9}{5^3} & 0 & \frac{12}{5^3} & \frac{54}{5^4} & \frac{72}{5^5} & 0 & \frac{12}{5^4} & \frac{108}{5^5} & \frac{72}{5^5} & \frac{18}{5^5} \\
0 & 0 & 0 & \frac{3}{5^2} & \frac{12}{5^3} & \frac{18}{5^4} & \frac{14}{5^3} & \frac{108}{5^4} & \frac{72}{5^4} & \frac{96}{5^5} & \frac{24}{5^4} & \frac{12}{5^3} & \frac{342}{5^5} & \frac{1008}{5^6} & \frac{252}{5^6} \\
0 & 0 & 0 & \frac{1}{5} & \frac{4}{5^2} & \frac{6}{5^3} & \frac{2}{5^2} & \frac{4}{5^2} & \frac{72}{5^4} & \frac{96}{5^5} & \frac{2}{5^3} & \frac{36}{5^4} & \frac{234}{5^5} & \frac{144}{5^5} & \frac{36}{5^5} \\
0 & 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5^2} & 0 & \frac{4}{5^2} & \frac{18}{5^3} & \frac{24}{5^4} & 0 & \frac{4}{5^3} & \frac{36}{5^4} & \frac{24}{5^4} & \frac{6}{5^4} \\
0 & 0 & 0 & 0 & 0 & \frac{3}{5} & 0 & 0 & \frac{6}{5^2} & \frac{8}{5^3} & 0 & 0 & \frac{6}{5^3} & \frac{24}{5^4} & \frac{6}{5^4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{5^2} & \frac{24}{5^3} & \frac{72}{5^4} & \frac{96}{5^5} & \frac{9}{5^3} & \frac{96}{5^4} & \frac{504}{5^5} & \frac{288}{5^5} & \frac{72}{5^5} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{6}{5^2} & \frac{18}{5^3} & \frac{24}{5^4} & \frac{1}{5^2} & \frac{14}{5^3} & \frac{81}{5^4} & \frac{48}{5^4} & \frac{12}{5^4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{6}{5^2} & \frac{8}{5^3} & 0 & \frac{2}{5^2} & \frac{3}{5^2} & \frac{48}{5^4} & \frac{12}{5^4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{4}{5^2} & 0 & 0 & \frac{3}{5^2} & \frac{12}{5^3} & \frac{3}{5^3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{5} & 0 & 0 & 0 & \frac{4}{5^2} & \frac{1}{5^2}
\end{pmatrix}
$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(3,5) process

3	4	5	5	6	7	6	7	8	9	7	8	9	10	11
2	3	4	4	5	6	5	6	7	8	6	7	8	9	10
1	2	3	3	4	5	4	5	6	7	5	6	7	8	9
0	2	3	3	4	5	4	5	6	7	5	6	7	8	9
0	1	2	2	3	4	3	4	5	6	4	5	6	7	8
0	0	1	0	2	3	0	3	4	5	0	4	5	6	7
0	0	0	2	3	4	3	4	5	6	4	5	6	7	8
0	0	0	1	2	3	2	3	4	5	3	4	5	6	7
0	0	0	0	1	2	0	2	3	4	0	3	4	5	6
0	0	0	0	0	1	0	0	2	3	0	0	3	4	5
0	0	0	0	0	0	2	3	4	5	3	4	5	6	7
0	0	0	0	0	0	1	2	3	4	2	3	4	5	6
0	0	0	0	0	0	0	1	2	3	0	2	3	4	5
0	0	0	0	0	0	0	0	1	2	0	0	2	3	4
0	0	0	0	0	0	0	0	0	1	0	0	0	2	3

Transition probability matrix of the semi-Markov generated point process equivalent to the E(5,2) process

$$\begin{pmatrix} \frac{1}{2^4} & \frac{5}{2^5} & \frac{15}{2^6} & \frac{35}{2^7} & \frac{35}{2^7} \\ \frac{1}{2^4} & \frac{5}{2^5} & \frac{15}{2^6} & \frac{35}{2^7} & \frac{35}{2^7} \\ \frac{1}{2^3} & \frac{3}{2^4} & \frac{7}{2^5} & \frac{15}{2^6} & \frac{15}{2^6} \\ \frac{1}{2^2} & \frac{1}{2^2} & \frac{3}{2^4} & \frac{5}{2^5} & \frac{5}{2^5} \\ \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \frac{1}{2^4} \end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(5,2) process

$$\begin{pmatrix} 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(5,3) process

$$
\begin{pmatrix}
\frac{1}{3^4} & \frac{10}{3^5} & \frac{10}{3^5} & \frac{70}{3^7} & \frac{140}{3^8} & \frac{10}{3^5} & \frac{70}{3^6} & \frac{560}{3^8} & \frac{140}{3^7} & \frac{140}{3^7} & \frac{280}{3^7} & \frac{700}{3^8} & \frac{1400}{3^9} & \frac{7700}{3^{10}} & \frac{3850}{3^{10}} \\
\frac{1}{3^4} & \frac{10}{3^5} & \frac{10}{3^5} & \frac{70}{3^7} & \frac{140}{3^8} & \frac{10}{3^5} & \frac{70}{3^6} & \frac{560}{3^8} & \frac{140}{3^7} & \frac{140}{3^7} & \frac{280}{3^7} & \frac{700}{3^8} & \frac{1400}{3^9} & \frac{7700}{3^{10}} & \frac{3850}{3^{10}} \\
\frac{1}{3^3} & \frac{2}{3^3} & \frac{14}{3^5} & \frac{10}{3^5} & \frac{20}{3^6} & \frac{4}{3^4} & \frac{70}{3^6} & \frac{20}{3^5} & \frac{140}{3^7} & \frac{40}{3^6} & \frac{700}{3^8} & \frac{1820}{3^9} & \frac{1120}{3^9} & \frac{700}{3^8} & \frac{350}{3^8} \\
\frac{1}{3^2} & \frac{4}{3^3} & \frac{2}{3^3} & \frac{10}{3^5} & \frac{20}{3^6} & \frac{2}{3^3} & \frac{8}{3^4} & \frac{50}{3^6} & \frac{40}{3^6} & \frac{10}{3^5} & \frac{50}{3^6} & \frac{140}{3^7} & \frac{70}{3^7} & \frac{140}{3^7} & \frac{70}{3^7} \\
\frac{1}{3} & \frac{2}{3^2} & \frac{2}{3^3} & \frac{2}{3^4} & \frac{4}{3^5} & \frac{2}{3^3} & \frac{2}{3^3} & \frac{8}{3^5} & \frac{20}{3^6} & \frac{2}{3^4} & \frac{20}{3^6} & \frac{20}{3^6} & \frac{20}{3^7} & \frac{140}{3^8} & \frac{70}{3^8} \\
0 & \frac{2}{3^4} & \frac{8}{3^5} & \frac{20}{3^6} & \frac{40}{3^7} & \frac{1}{3^3} & \frac{70}{3^6} & \frac{190}{3^7} & \frac{140}{3^7} & \frac{50}{3^6} & \frac{910}{3^8} & \frac{2240}{3^9} & \frac{1540}{3^9} & \frac{2800}{3^9} & \frac{1400}{3^9} \\
0 & \frac{1}{3^3} & \frac{4}{3^4} & \frac{10}{3^5} & \frac{20}{3^6} & \frac{1}{3^3} & \frac{23}{3^5} & \frac{65}{3^6} & \frac{50}{3^6} & \frac{5}{3^4} & \frac{275}{3^7} & \frac{700}{3^8} & \frac{455}{3^8} & \frac{280}{3^7} & \frac{140}{3^7} \\
0 & \frac{1}{3^2} & \frac{2}{3^3} & \frac{4}{3^4} & \frac{8}{3^5} & \frac{2}{3^3} & \frac{1}{3^2} & \frac{7}{3^4} & \frac{50}{3^6} & \frac{4}{3^4} & \frac{65}{3^6} & \frac{20}{3^5} & \frac{95}{3^7} & \frac{560}{3^8} & \frac{280}{3^8} \\
0 & \frac{1}{3} & \frac{1}{3^2} & \frac{1}{3^3} & \frac{2}{3^4} & \frac{1}{3^2} & \frac{1}{3^2} & \frac{4}{3^4} & \frac{10}{3^5} & \frac{1}{3^3} & \frac{10}{3^5} & \frac{10}{3^5} & \frac{10}{3^6} & \frac{70}{3^7} & \frac{35}{3^7} \\
0 & 0 & \frac{2}{3^3} & \frac{2}{3^3} & \frac{4}{3^4} & 0 & \frac{2}{3^3} & \frac{8}{3^4} & \frac{20}{3^5} & \frac{13}{3^5} & \frac{10}{3^4} & \frac{80}{3^6} & \frac{50}{3^6} & \frac{280}{3^7} & \frac{140}{3^7} \\
0 & 0 & \frac{1}{3^2} & \frac{1}{3^2} & \frac{2}{3^3} & 0 & \frac{2}{3^3} & \frac{1}{3^2} & \frac{8}{3^4} & \frac{1}{3^3} & \frac{23}{3^5} & \frac{70}{3^6} & \frac{35}{3^6} & \frac{70}{3^6} & \frac{35}{3^6} \\
0 & 0 & \frac{1}{3} & \frac{1}{3^2} & \frac{2}{3^3} & 0 & \frac{1}{3^2} & \frac{2}{3^3} & \frac{2}{3^3} & \frac{1}{3^3} & \frac{4}{3^4} & \frac{14}{3^5} & \frac{4}{3^5} & \frac{5}{3^5} & \frac{10}{3^5} \\
0 & 0 & 0 & \frac{2}{3^2} & \frac{4}{3^3} & 0 & 0 & \frac{4}{3^3} & \frac{4}{3^3} & 0 & \frac{2}{3^3} & \frac{8}{3^4} & \frac{1}{3^3} & \frac{20}{3^5} & \frac{10}{3^5} \\
0 & 0 & 0 & \frac{1}{3} & \frac{2}{3^2} & 0 & 0 & \frac{1}{3^2} & \frac{4}{3^3} & 0 & \frac{1}{3^3} & \frac{2}{3^3} & \frac{1}{3^4} & \frac{10}{3^5} & \frac{5}{3^5} \\
0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{2}{3^2} & 0 & 0 & \frac{2}{3^3} & 0 & \frac{2}{3^4} & \frac{1}{3^4}
\end{pmatrix}
$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(5,3) process

$$\begin{pmatrix}
5 & 6 & 7 & 8 & 9 & 7 & 8 & 9 & 10 & 9 & 10 & 11 & 11 & 12 & 13 \\
4 & 5 & 6 & 7 & 8 & 6 & 7 & 8 & 9 & 8 & 9 & 10 & 10 & 11 & 12 \\
3 & 4 & 5 & 6 & 7 & 5 & 6 & 7 & 8 & 7 & 8 & 9 & 9 & 10 & 11 \\
2 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 7 & 6 & 7 & 8 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & 6 & 5 & 6 & 7 & 7 & 8 & 9 \\
0 & 4 & 5 & 6 & 7 & 5 & 6 & 7 & 8 & 7 & 8 & 9 & 9 & 10 & 11 \\
0 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 7 & 6 & 7 & 8 & 8 & 9 & 10 \\
0 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & 6 & 5 & 6 & 7 & 7 & 8 & 9 \\
0 & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 4 & 5 & 6 & 6 & 7 & 8 \\
0 & 0 & 3 & 4 & 5 & 0 & 4 & 5 & 6 & 5 & 6 & 7 & 7 & 8 & 9 \\
0 & 0 & 2 & 3 & 4 & 0 & 3 & 4 & 5 & 4 & 5 & 6 & 6 & 7 & 8 \\
0 & 0 & 1 & 2 & 3 & 0 & 2 & 3 & 4 & 3 & 4 & 5 & 5 & 6 & 7 \\
0 & 0 & 0 & 2 & 3 & 0 & 0 & 3 & 4 & 0 & 4 & 5 & 5 & 6 & 7 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 3 & 0 & 3 & 4 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 4 & 5
\end{pmatrix}$$

Transition probability matrix of the semi-Markov generated point process equivalent to the E(8,2) process

$$\begin{pmatrix}
\frac{1}{2^7} & \frac{1}{2^5} & \frac{9}{2^7} & \frac{15}{2^7} & \frac{165}{2^{10}} & \frac{99}{2^9} & \frac{429}{2^{11}} & \frac{429}{2^{11}} \\
\frac{1}{2^7} & \frac{1}{2^5} & \frac{9}{2^7} & \frac{15}{2^7} & \frac{165}{2^{10}} & \frac{99}{2^9} & \frac{429}{2^{11}} & \frac{429}{2^{11}} \\
\frac{1}{2^6} & \frac{3}{2^6} & \frac{11}{2^7} & \frac{1}{2^3} & \frac{81}{2^9} & \frac{93}{2^9} & \frac{99}{2^9} & \frac{99}{2^9} \\
\frac{1}{2^5} & \frac{5}{2^6} & \frac{15}{2^7} & \frac{9}{2^6} & \frac{39}{2^8} & \frac{81}{2^9} & \frac{165}{2^{10}} & \frac{165}{2^{10}} \\
\frac{1}{2^4} & \frac{1}{2^3} & \frac{5}{2^5} & \frac{5}{2^5} & \frac{9}{2^6} & \frac{1}{2^3} & \frac{15}{2^7} & \frac{15}{2^7} \\
\frac{1}{2^3} & \frac{3}{2^4} & \frac{3}{2^4} & \frac{5}{2^5} & \frac{15}{2^7} & \frac{11}{2^7} & \frac{9}{2^7} & \frac{9}{2^7} \\
\frac{1}{2^2} & \frac{1}{2^2} & \frac{3}{2^4} & \frac{1}{2^3} & \frac{5}{2^6} & \frac{3}{2^6} & \frac{1}{2^5} & \frac{1}{2^5} \\
\frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \frac{1}{2^5} & \frac{1}{2^6} & \frac{1}{2^7} & \frac{1}{2^7}
\end{pmatrix}$$

Transition density shape parameter matrix of the semi-Markov point process equivalent to the E(8,2) process

$$\begin{pmatrix}
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{pmatrix}$$

APPENDIX D

Serial correlations, $\{\rho_j\}_{j=1}^{n}$ and damping factors $\{\alpha_i\}_{i=1}^{n}$ for selected E(k,p) processes. Each E(k,p) process has a similar ARMA(m,n) process, with m and n estimated in Figure III.C.1. For indices greater than n, the serial correlation is given by

$$\rho_j = \rho_{j-1}\alpha_1 + \ldots + \rho_{j-m}\alpha_m , \qquad j>n .$$

Those processes marked with a single asterisk had correlations computed from their semi-Markov generated point process representation using (5.B.3) and (5.B.4). Correlations for the remaining processes were determined from a fast Fourier transform of the interval spectrum as determined by SPECD. (VIII.A.1 and VIII.E)

Processes marked with a double asterisk have parameters m or n less than indicated in Figure III.C.1.

Process: E(2,2)

j	ρ_j
1	-0.1
2,3,...	0

Similar Process: ARMA(0,1)

i	α_i
2,3,...	0

$|\epsilon| = 0$

Process: E(2,3)*

j	ρ_j
1	-0.10444
2	-0.03160

Similar Process: ARMA(1,2)

i	α_i
1	.11111 (1/9)

$|\varepsilon| < 10^{-15}$

Process: E(2,4)*

j	ρ_j
1	-0.09796
2	-0.04428
3	-0.01503

Similar Process: ARMA(1,3)

i	α_i
1	.25 (1/4)

$|\varepsilon| < 10^{-15}$

Process: E(2,5)*

j	ρ_j
1	-0.09022
2	-0.04898
3	-0.02225
4	-0.00925

Similar Process: ARMA(2,4)

i	α_i
1	-0.01434
2	0.4

$|\varepsilon| < 10^{-15}$

Process: E(2,6)

j	ρ_j
1	-0.08303
2	-0.05022
3	-0.02773
4	-0.01398
5	-0.00655

Similar Process: ARMA(2,5)

i	α_i
1	-.05004
2	.55697

$|\varepsilon| < 10^{-9}$

Process: E(3,2)*

j	ρ_j
1	-0.20341
2	-0.03452

Similar Process: ARMA(1,2)

i	α_i
1	-0.125 (1/8)

$|\varepsilon| < 10^{-15}$

Process: E(3,3)*

j	ρ_j
1	-0.18847
2	-0.05455

Similar Process: ARMA(2,4)

i	α_i
1	-0.03704
2	1.6×10^{-12}

$|\varepsilon| < 10^{-14}$

Process: E(3,4)

j	ρ_j
1	-0.16469
2	-0.08044
3	-0.01731
4	0.00746
5	0.00310
6	0.00044

Similar Process: ARMA(4,6)

i	α_i
1	-0.00426
2	-0.02583
3	0.02192
4	-0.18688

$|\varepsilon| < 10^{-8}$

Process: E(3,5)*

j	ρ_j
1	-0.14442
2	-0.08673
3	-0.03744
4	-0.00656
5	0.00392
6	0.00270
7	0.00099
8	0.00018

Similar Process: ARMA(6,8)**

i	α_i
1	0.0000025
2	0.00016
3	-0.00251
4	-0.00681
5	-0.09253
6	0.45096

$|\varepsilon| < 10^{-18}$

Process: E(4,2)

j	ρ_j
1	-0.29238
2	0.83825

Similar Process: ARMA(1,2)

i	α_i
1	-0.25 (1/4)

$|\varepsilon| < 10^{-6}$

Process: E(4,3)

j	ρ_j
1	-0.24870
2	-0.07548
3	0.04852
4	-0.00118
5	-0.00442

Similar Process: ARMA(3,5)

i	α_i
1	-0.02464
2	-0.14004
3	-0.43167

$|\varepsilon| < 10^{-7}$

Process: E(4,4)

j	ρ_j
1	-0.20728
2	-0.11047
3	-0.01699
4	0.02770
5	0.00493
6	-0.00223
7	-0.00147
8	0.00008

Similar Process: ARMA(5,8)

i	α_i
1	0.02325
2	0.07804
3	0.02958
4	0.52518
5	0.46057

$|\varepsilon| < 10^{-10}$

Process: E(5,2)*

j	ρ_j
1	-0.36619
2	0.13681
3	-0.04711

Similar Process: ARMA(2,3)

i	α_i
1	0.000977
2	-0.3437

$|\varepsilon| < 10^{-14}$

Process: E(5,3)*

j	ρ_j
1	-0.29176
2	-0.09656
3	0.08688
4	-0.01020
5	-0.01068
6	0.00478
7	0.00028
8	-0.00084

Similar Process: ARMA(6,8)

i	α_i
1	0.0000158
2	0.000874
3	-0.00423
4	-0.87306
5	-1.80986
6	-5.75229

$|\varepsilon| < 10^{-19}$

Process: E(6,2)

j	ρ_j
1	-0.42728
2	0.18853
3	-0.07884

Similar Process: ARMA(2,3)

i	α_i
1	0.00613
2	-0.40735

$|\varepsilon| < 10^{-6}$

Process: E(6,3)

j	ρ_j
1	-0.32458
2	-0.11478
3	0.12700
4	-0.02400
5	-0.01846
6	0.01257
7	-0.00084
8	-0.00245

Similar Process: ARMA(6,8)

i	α_i
1	0.02899
2	-0.43110
3	-0.01320
4	-0.02046
5	-0.80478
6	-0.04981

$|\varepsilon| < 10^{-6}$

Process: E(8,2)*

j	ρ_j
1	-0.52123
2	0.28153
3	-0.14813
4	-0.07870

Similar Process: ARMA(3,4)

i	α_i
1	0.00001526
2	0.03296
3	-0.46875

$|\varepsilon| < 10^{-14}$

E(K,P) SPECTRUM BY NUMERICAL APPROXIMATION - SPECD

PROGRAMMED BY W. J. HAYNE

```
C       CALLING ARGUMENTS- F IS AN ARRAY OF POINT EXTIMATES OF
                           THE SPECTRAL DENSITY OF AN ERLANG SUPERPOSITION
                           (RETURNED BY SPECD)
C                          NDIM IS THE NUMBER OF POINTS TO BE ESTIMATED
                           KC IS THE SHAPE PARAMETER OF THE COMPONENT PROCESS
                           IP IS THE NUMBER OF COMPONENTS
                           NPRINT = 1   - PRINT DETAILED COMPUTATION

      SUBROUTINE SPECD ( F, NDIM,KC,IP,NPRINT)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION ANG(10), F(NDIM), JMAT(1061,15),RN(10),AI(10)
     1,NFAC(13),NX(10)                                                  SPE00020
      DATA PI/ 3.14159265358979D0/
      PI2=2.D0*PI
      PII=1.D0/PI
      NFAC(1)=1                                                         SPE00060
      DO 210 N=1,12                                                     SPE00070
      NP=N+1                                                            SPE00080
      NFAC(NP)=N*NFAC(N)                                                SPE00090
210   CONTINUE                                                          SPE00100
      DO 10 J=1,10                                                      SPE00110
      ANG(J)=PI2*DFLOAT(J)
10    CONTINUE                                                          SPE00130
      IP1=IP+1
      AIP = DFLOAT(IP)
      KPM1=KC+IP                                                        SPE00170
      KM1=KC-1                                                          SPE00180
      MJMAT=NFAC(KPM1)/(NFAC(IP1)*NFAC(KC))                             SPE00190
      WRITE (6,22) KC,IP,MJMAT                                          SPE00200
22    FORMAT ('1', 5X,'K=',I2,',  P=',I2,',  NO. COMB.=',I3,'.')        SPE0
      IP1=IP+1                                                          SPE00220
      AKC = DFLOAT(KC)
      DO 220 J=1,IP1                                                    SPE00240
      JMAT(1,J)=1                                                       SPE00250
220   CONTINUE                                                          SPE00260
      DO 300 I=2,MJMAT                                                  SPE00280
      IM=I-1                                                            SPE00290
      DO 230 J=1,IP                                                     SPE00300
      JMAT(I,J)=JMAT(IM,J)                                              SPE00310
230   CONTINUE                                                          SPE00320
      J=0                                                               SPE00330
232   J=J+1                                                             SPE00340
```

```
      JMJ=JMAT(I,J)                                                    SPE00350
      IF(JMJ.LT.KC) GO TO 244
  240 IF(J.LT.IP) GO TO 232
      GO TO 199                                                        SPE00380
  244 JMAT(I,J)=JMAT(I,J)+1                                            SPE00390
      IF(J.EQ.1) GO TO 252                                             SPE00400
      JM=J-1                                                           SPE00410
      DO 250 JA=1,JM                                                   SPE00420
      JMAT(I,JA)=JMAT(I,J)                                             SPE00430
  250 CONTINUE                                                         SPE00440
  252 DO 260 IX=1,KC                                                   SPE00450
      NX(IX)=1                                                         SPE00460
  260 CONTINUE                                                         SPE00470
      DO 270 JB=1,IP                                                   SPE00480
      IX=JMAT(I,JB)                                                    SPE00490
      NX(IX)=NX(IX)+1                                                  SPE00500
  270 CONTINUE                                                         SPE00510
      IPROD =1                                                         SPE00520
      DO 280 IY=1,KC                                                   SPE00530
      IN=NX(IY)                                                        SPE00540
      IPROD=IPROD*NFAC(IN)                                             SPE00550
  280 CONTINUE                                                         SPE00560
      JMAT(I,IP1)=NFAC(IP1)/IPROD                                      SPE00570
  300 CONTINUE                                                         SPE00600
      AKI2=1./AKC**2                                                   SPE00610
      SUM4=0.                                                          SPE00620
      NDIM1 = NDIM - 1
      AINC = PI / DFLOAT(NDIM1)
      NDIMM = NDIM - 2
      DO 100 I = 1, NDIMM
      OM=AINC*DFLOAT(I)
      CF1=(AKI2*(1.-DCOS(OM)))**IP
      CF1P=PI1*CF1                                                     SPE00680
      DO 30 J=1,KC                                                     SPE00690
      OMJ=(ANG(J)+OM)/AKC                                              SPE00700
      RN(J)=1.-DCOS(OMJ)
      AI(J)=DSIN(OMJ)
   30 CONTINUE                                                         SPE00730
      SUM3=0.                                                          SPE00740
      DO 50 MJ=1,MJMAT                                                 SPE00750
      J11=JMAT(MJ,IP1)                                                 SPE00760
      CF2 = DFLOAT(J11)
      SUM1=0.                                                          SPE00780
      SUM2=0.                                                          SPE00790
      PROD1=1.                                                         SPE00800
      DO 40 NJ=1,IP                                                    SPE00810
      J=JMAT(MJ,NJ)                                                    SPE00820
      SUM1=SUM1+RN(J)                                                  SPE00830
```

```
      SUM2=SUM2+AI(J)                                                    SPE00840
      PROD1=PROD1*RN(J)                                                  SPE00850
40    CONTINUE                                                           SPE00860
      ANORM=SUM1**2+SUM2**2                                              SPE00870
      ANUM=2.*SUM1                                                       SPE00880
      DENOM=ANORM*PROD1                                                  SPE00890
      IF(DENOM.EQ.0.) GO TO 129                                          SPE00900
      SUM3=SUM3 + CF2*ANUM/DENOM                                         SPE00910
50    CONTINUE                                                           SPE00920
      IPLUS1=I+1                                                         SPE00930
      F(IPLUS1)=CF1P*SUM3                                                SPE00940
      SUM4=SUM4+F(IPLUS1)                                                SPE00950
100   CONTINUE                                                           SPE00960
      F(1)=3.*(F(2)-F(3))+F(4)                                           SPE00970
      F (NDIM) = 3. * ( F (NDIM-1) - F (NDIM-2)) + F (NDIM-3)
      SUM4 = SUM4 + .5 * ( F (1) + F (NDIM))
      ALFA = DFLOAT(NDIM1) /(PI*SUM4)
      IF (NPRINT .EQ.1)
     2WRITE (6,101) KC,IP                                                SPE01010
101   FORMAT ('0    K=',I2,',   P=',I2,'.')                              SPE01020
      IF (NPRINT .EQ.1)
     2WRITE (6,109) ALFA                                                 SPE01030
109   FORMAT ('0 NORMALIZING FACTOR=', F13.6)                            SPE01040
      ALFAI=1.D0/ALFA
      RPK= AIP/ AKC
      CVX = DSQRT(RPK*ALFAI)
      VX=ALFAI/RPK                                                       SPE01080
      WRITE (6,103) CVX,VX                                               SPE01090
103   FORMAT ('0COEFFICIENT OF VAR.=',F13.6 / '  WHEN ',                 SPE01100
     1'LAMBDA EQUALS ONE, VAR(X)=', F13.6 )                              SPE01110
      DO 120 I = 1, NDIM
      F(I)=ALFA*F(I)                                                     SPE01130
      IF (NPRINT .EQ.1)
     2WRITE (6,111) I,F(I)                                               SPE01140
111   FORMAT ('0', 5X,I2, F13.6)                                         SPE01150
120   CONTINUE                                                           SPE01160
      RETURN
129   WRITE (6,131)                                                      SPE01180
131   FORMAT ('0 DENOM EQUALS ZERO.' )                                   SPE01190
  132 WRITE (6,133) I, OM                                                SPE01200
133   FORMAT ('0  I=',I2,',  OM=',F7.3 )                                 SPE01210
      WRITE (6,135) ANUM,ANORM,PROD1                                     SPE01220
135   FORMAT ('0ANUM=', F15.7,',   ANORM=',F15.7,',  PROD1=',F15.7)      SPE01230
   20 CONTINUE
  199 RETURN
  283 FORMAT (  ' JMAT ',10I10 )
      END                                                                SPE01250
```

```
          (SEMI-MARKOV REPRESENTATION)
C   PROGRAM TO DETERMINE THE SPECTRAL DENSITY OF A SEMI-MARKOV GENERATED
C   POINT PROCESS WITH GAMMA TRANSITION DENSITIES
    INPUT PARAMETERS
       N        THE DIMENSION OF THE STATE SPACE
       K        THE SHAPE PARAMETER OF THE COMPONENT ERLANG PROCESS
       NP       THE NUMBER OF COMPONENT PROCESSES
       NPRINT     A PARAMETER OF THE LINEAR EQUATIONS ROUTINE
       NPLOT    =1 IF A PLOT IS TO BE MADE
       NPTS,  MC, ITYPE, IX, IY, MDX, MDY, IW, IH, IG   PLOT PARAMETERS
       XSC, YSC     PLOT PARAMETERS
       TITLE     PLOT PARAMETER
       T        TRANSITION MATRIX(NUMERATORS)
       NTEXP     TRANSITION MATRIX (DENOMINATOR COEFFICIENTS)
       E        SHAPE PARAMETER MATRIX

    SUBROUTINES CALLED
       LINEM    SOLVES LINEAR SYSTEM OF EQUATIONS
       DRAW     PLOT ROUTINE

       REAL*4 M1,M2
       REAL*4 LABEL,X(512),Y(512)
       REAL*8 TITLE(12)
       DIMENSION  T(16,16), E(16,16), M1(16,16),M2(16,16), PI(16),VL(16)
       DIMENSION  VR (16), NTEXP(16,16)
       DIMENSION B(16,16)
       DATA LABEL/'    '/
       FORMAT OPTIONS
     2 CONTINUE
       MC=10
       READ (5,500,END=1) N,K,NP,NPRINT,NPLOT
       WRITE(6,500) N,K,NP,NPRINT,NPLOT
       P=FLOAT(NP)
       LET (N='N')
       IF(NPLOT.EQ.1)READ (5,500)NPTS,MC,ITYPE,IX,IY,MDX,MDY,IW,IH,IG
       IF(NPLOT.EQ.1)WRITE(6,500)NPTS,MC,ITYPE,IX,IY,MDX,MDY,IW,IH,IG
       IF(MC.LE.1)READ (5,501)XSC,YSC
       IF(MC.LE.1)WRITE(6,503)XSC,YSC
       IF(MC.LE.1)READ (5,502)(TITLE(I),I=1,12)
       IF(MC.LE.1)WRITE(6,502)(TITLE(I),I=1,12)
   502 FORMAT(6A8)
       IF(K.EQ.-1)GO TO 300
       DO 100 I = 1, N
       READ (5,501) ( T(I,J),J=1,N)
```

```
      WRITE(6,501) ( T(I,J),J=1,N)
      READ (5,500) ( NTEXP (I,J),J=1,N)
      WRITE(6,500) ( NTEXP (I,J),J=1,N)
  100 CONTINUE
      DO 110 I=1, N
      READ (5,501) ( E(I,J),J=1,N )
      WRITE(6,501) ( E(I,J),J=1,N )
  110 CONTINUE
  500 FORMAT (16I5)
  501 FORMAT (16F5.0)
      DO 120 I=1, N
      DO 120 J=1, N
      NPWR=NTEXP(I,J)
      T(I,J)=T(I,J)/P**NPWR
      M1 (I,J) = T(I,J)* E(I,J)/ P
      M2 (I,J) = M1(I,J)*(1+E(I,J))/P
  120 CONTINUE
      GO TO 320
  300 CONTINUE
      READ (5,501)Q,R
      WRITE(6,503)Q,R
  503 FORMAT (12F10.0)
      T(1,1)=Q
      T(1,2)=1.-Q
      T(1,3)=0.
      T(2,1)=Q-R*Q
      T(2,2)=1.-R-Q+2.*R*Q
      T(2,3)=R-R*Q
      T(3,1)=0.
      T(3,2)=Q
      T(3,3)=1.-Q
      M1(1,1)=Q/(2.*R)
      M1(1,2)=(1.-Q)/(2.*R)
      M1(1,3)=0.
      M1(2,1)=T(2,1)
      M1(2,2)=T(2,2)
      M1(2,3)=T(2,3)
      M1(3,1)=0.
      M1(3,2)=Q/(2.-2.*R)
      M1(3,3)=(1.-Q)/(2.-2.*R)
      M2(1,1)=M1(1,1)/R
      M2(1,2)=M1(1,2)/R
      M2(1,3)=0.
      M2(2,1)=2.*M1(2,1)
      M2(2,2)=2.*M1(2,2)
      M2(2,3)=2.*M1(2,3)
      M2(3,1)=0.
      M2(3,2)=M1(3,2)/(1.-R)
```

```
      M2(3,3)=M1(3,3)/(1.-R)
      WRITE(6,610)((T(I,J),J=1,3),I=1,3)
      WRITE(6,611)((M1(I,J),J=1,3),I=1,3)
      WRITE(6,612)((M2(I,J),J=1,3),I=1,3)
  610 FORMAT('          HE TRANSITION MATRIX'/3E21.7/3E21.7/3E21.7)
  611 FORMAT(' WEIGHTED EXPECTATION MATRIX'/3E21.7/3E21.7/3E21.7)
  612 FORMAT('  SQUARED EXPECTATION MATRIX'/3E21.7/3E21.7/3E21.7)
  320 CONTINUE
      DO 130 I=1,N
      DO 130 J=1,N
      NPWR=NTEXP(J,I)
      B(I,J)= T(J,I)
      IF (I.EQ.J) B(I,J)= B(I,J)-1.
  130 CONTINUE
      DO 140 J=1, N
      PI (J)= 0.
      B( N,J) = 1.
  140 CONTINUE
      PI (N) = 1.
      CALL LINEM ( B, PI, N, NPRINT )
      WRITE ( 6,601) ( PI(I), I=1,N)
  601 FORMAT (/ ' STATIONARY  VECTOR', / 8E16.7,/8E16.7 // )
      V=0.
      EV=0.
      DO 150 J=1,N
      VL(J)=0.
      VR(J)=0.
      DO 150 I=1,N
      VL(J)=VL(J)+PI(I)*M1(I,J)
      VR(J)=VR(J)+M1(J,I)
      V = V + PI(I)* M2(I,J)
      EV=EV+PI(I)*M1(I,J)
  150 CONTINUE
      DO 160 I=1,N
      LET ( I='I')
      DO 160 J=1,N
      LET (J='J')
      U =-T(I,J)
      LET ( A(I,J)= 'U')
      LET(B(I,J)=0)
  160 CONTINUE
      DO 170 I=1,N
      LET (I='I')
       LET(U=X+A(I,I)    )
      LET(A(I,I)=U)
      LET(B(I,I)=1)
  170 CONTINUE
      ATOMIZE (U)
```

```
      DO 50 I = 1, N
      WRITE(6,607)I
  607 FORMAT ('    START INVERSION STEP',I2)
      LET (I='I')
      LET (F = A(I,I))
      LET(COF(I)=F)
      KEND = I-1
      IF (KEND .EQ. 0 ) GO TO 31
      LET(KM='KEND')
      LET(   DD=A(KM,KM))
      LET(ND=HIGHPOW(DD,X))
      ND= INTEGER(ND)
      LET(SUM=0)
      LET (H=DD)
      DO 25NI=1,ND
      LET(NJ='ND'+2-'NI')
      LET(NK=NJ-1)
      LET(ALFA(NJ)=COEFF(H,X**NK))
      LET(U=ALFA(NJ)*X**NK+SUM)
      LET(SUM=U)
   25 CONTINUE
      WRITE(6,606)
  606 FORMAT('   COMPLETED PHASE ONE')
      LET(ALFA(1)=H-SUM)
      DO 30 L=1,KEND
      LET (K='L')
      LET (G=A(K,I))
      DO 30 J = 1, N
      LET ( J = 'J' )
      LET(H=F*A(K,J)-G*A(I,J))
      LET (T=EXPAND(H)   )
      LET(H=T)
      IRET=1
      GO TO 45
   29 CONTINUE
      LET(A(K,J)=SUM)
      LET(H=F*B(K,J)-G*B(I,J))
      LET(T=EXPAND(H))
      LET(H=T)
      IRET=2
      GO TO 45
   32 CONTINUE
      LET(B(K,J)=SUM)
   30 CONTINUE
   31 CONTINUE
      WRITE(6,608)
  608 FORMAT('    COMPLETED PHASE TWO')
      KS = I+1
```

```
      IF ( KS .GT. N ) GO TO 50
      DO 40 L = KS,N
      LET (K='L')
      LET (G=A(K,I))
      DO 40 J = 1, N
      LET ( J ='J' )
      LET(H=F*A(K,J)-G*A(I,J))
      LET(A(K,J)=EXPAND(H))
      LET(H=F*B(K,J)-G*B(I,J))
      LET(B(K,J)=EXPAND(H))
      IF(I.EQ.1) GO TO 40
      LET( H=A(K,J))
      IRET=3
      GO TO 45
   34 CONTINUE
      LET(A(K,J)=SUM)
      LET(H=B(K,J))
      IRET=4
      GO TO 45
   36 CONTINUE
      LET(B(K,J)=SUM)
   40 CONTINUE
   50 CONTINUE
      NEXP=100
      TST=0.0
      DO 57 J=1,N
      LET(J='J')
      PRINT OUT(COF(J))
      DO 57 L=1,N
      LET(K='L')
      PRINT OUT(B(J,K))
      IF (TST.NE.0.)          GO TO 57
      LET (TST=REPLACE(B(J,K),X,0.0))
      TST=ARITH(TST)
      LET(S=LOWPOW(B(J,K),X))
      NTXP=INTEGER(S)
      IF (NEXP.GT.NTXP)NEXP=NTXP
   57 CONTINUE
      IF (TST.NE.0.0) GO TO 61
      LET (TST=REPLACE(A(N,N),X,0.0))
      TST=ARITH(TST)
      IF (TST.NE.0.0) GO TO 61
      LET(S=LOWPOW(A(N,N),X))
      NTXP=INTEGER(S)
      IF (NEXP.GT.NTXP)NEXP=NTXP
      IF(NEXP.LE.0)GO TO 61
      LET(NEXP='NEXP')
      DO 55 I=1,N
```

```
      LET (I='I')
      DO 55 J=1,N
      LET(J='J')
      LET(U=B(I,J)/X**NEXP)
      LET(B(I,J)=EXPAND(U))
   55 CONTINUE
      LET(U=COF(N)/X**NEXP)
      LET(COF(N)=EXPAND(U))
      GO TO 61
   45 CONTINUE
      LET (SUM=0)
      LET(Z= REPLACE(H,X,1.629))
      H=ARITH(Z)
      IF (H.EQ.0.) GO TO 370
      LET(MD=HIGHPOW(H,X))
      MD=INTEGER (MD)
      LET(SUM=0)
      DO 35 NI=1,MD
      LET(NJ='MD'+2-'NI')
      LET(NK=NJ-1)
      LET(BETA(NJ)=     COEFF(H,X**NK))
      LET(U=BETA(NJ)*X**NK+SUM)
      LET(SUM=U)
   35 CONTINUE
      LET(BETA(1)=H-SUM)
      IF (IRET.GE.5)GO TO 85
      LET(SUM=0)
       JMAX=MD-ND+1
      DO 37 JN=1,JMAX
      LET(JN='JN')
      LET(KJ='JMAX'-'JN'+1)
      LET(FCT=BETA(MD+2-JN)/ALFA(ND+1))
      LET(U=FCT*X**(KJ-1)+SUM)
      LET(SUM=U)
      DO 38 KN=1,ND
      LET(KN='KN')
      LET(BETA(MD+2-JN-KN)=BETA(MD+2-JN-KN)-FCT*ALFA(ND-KN+1)     )
   38 CONTINUE
   37 CONTINUE
  370 CONTINUE
      GO TO (29,32,34,36,85,85),IRET
   61 CONTINUE
   59 CONTINUE
      DO 60   J=1,N
      LET ( J='J')
      UL=VL(J)
      UR=VR(J)
      LET ( VL(J)='UL')
```

```
      LET ( VR(J)='UR')
60    CONTINUE
      LET (T=0)
      DO 70 J=1,N
      LET ( J='J' )
      LET ( W(J)= 0 )
      DO 70 I=1,N
      LET ( I='I')
      LET ( S= VL(I)*B(I,J)+ W(J))
      LET(W(J)=EXPAND(S))
70    CONTINUE
      LET(T=0)
      DO 80 I=1,N
      LET(I='I')
      LET (S=W(I)*VR(I)+T)
      LET(T=EXPAND(S))
80    CONTINUE
      LET(H=COF(N))
      IRET=5
      GO TO 45
85    CONTINUE
      IF (IRET.EQ.6) GO TO 87
      MU=MD  +1
      DO 86 I=1,MU
      LET(I='I')
      LET(ALFA (I)=BETA(I))
86    CONTINUE
      LET(H=T)
      IRET=6
      GO TO 45
87    CONTINUE
      LET(SUM=0)
      MM=MU-1
      DO 90 I=1,MM
      IP=I+1
      LET(I='I')
      LET(U=SUM+ALFA(I)**2)
      LET(SUM=U)
      DO 90 J=IP,MU
      LET(J='J')
      LET(U=SUM+2*ALFA(I)*ALFA(J)*C(J-I))
      LET(SUM=U)
90    CONTINUE
      LET(MU='MU')
      LET(U=SUM+ALFA(MU)**2)
      LET(D=U)
      LET (SUM=0)
      LET(C(0)=1)
```

```
      MD=MD+1
      DO 95 I=1,MU
      LET(I='I')
      DO 95 J=1,MD
      LET(J='J')
      KK=IABS(I-J)
      LET(KK='KK')
      LET(U=SUM+2*ALFA(I)*BETA(J)*C(KK))
      LET(SUM=U)
   95 CONTINUE
      LET(V='V')
      LET (MEAN='EV')
      LET(SIG=V-MEAN**2)
      LET(U=SUM+D*V)
      LET(SUM=EXPAND(U))
      LET(U=SUM/(3.141592*SIG))
      LET(NMR=EXPAND(U))
      PRINTOUT(MEAN)
      PRINT OUT(SIG)
      PRINT OUT(CVSQR=SIG/MEAN**2)
      PRINT OUT(DNM=D)
      PRINT OUT (NMR)
      IF (NPLOT.EQ.0)GO TO 2
      DELTA=3.141592/FLOAT(NPTS)
      KMAX=MAXO(MU-1,MD-1)
      DO 210 I=1,NPTS
      THETA=DELTA*FLOAT(I)
      X(I)=THETA
      LET(SUMN=NMR)
      LET(SUMD=DNM)
      DO 200 J=1,KMAX
      LET(J='J')
      OMEGA=THETA*FLOAT(J)
      CS=COS(OMEGA)
      LET(G(J)='CS')
      LET (U=REPLACE(SUMN,C(J),G(J)))
      LET(SUMN=U)
      LET(U=REPLACE(SUMD,C(J),G(J)))
      LET(SUMD=U)
  200 CONTINUE
      LET(Y=SUMN/SUMD)
      Y(I)=ARITH(Y)
  210 CONTINUE
      WRITE(6,        603)
  603 FORMAT('     SPECTRAL DENSITY')
      BLANK=0.
      WRITE(6,604)BLANK,(Y(I),I=1,NPTS)
  604 FORMAT(8E16.7)
```

```
      CALL DRAW(NPTS,X,Y,MC,ITYPE,LABEL,TITLE,XSC,YSC,IX,IY,MDX,MDY,IW,
     2          IH,IG,LAST)
      WRITE(6,602)LAST
      GO TO 2
    1 CONTINUE
      STOP
  602 FORMAT('      LAST=',I2)
      END
```

```
          (SEMI-MARKOV REPRESENTATION)
C   PROGRAM TO DETERMINE THE SPECTRAL DENSITY OF A SEMI-MARKOV GENERATED
C   POINT PROCESS WITH GAMMA TRANSITION DENSITIES

    INPUT PARAMETERS
       N       THE DIMENSION OF THE STATE SPACE
       K       THE SHAPE PARAMETER OF THE COMPONENT PROCESS
       NP      THE NUMBER OF COMPONENT PROCESSES
       NPRINT - AN INPUT PARAMETER FOR THE LINEAR EQUATIONS ROUTINE
       T       TRANSITION MATRIX(NUMERATORS)
       NTEXP    TRANSITION MATRIX (DENOMINATOR COEFFICIENTS)
       E       SHAPE PARAMETER MATRIX

    SUBROUTINES CALLED
       DLINE   SOLVES LINEAR EQUATIONS
       DRAW    PLOT ROUTINE
       DEIGENP    COMPUTES EIGENVALUES AND EIGENVECTORS (IBM)
       DCMXIN    COMPUTES INVERSE OF A COMPLEX MATRIX (IBM)

      CMULT, CVLMLT, CMSUB, CVRMLT INCLUDED WITH THIS LISTING

       COMPLEX*16 COMP
       REAL*8 CMEGA
       REAL*8 EMPTY,P,R,C,TS,V,SUM,VAR,DIF,VRR,VRI,VLR,VLI
       REAL*8     A(16,16),EVR(16),EVI(16),VECR(16,16),VECI(16,16)
       REAL*8      T(16,16), E(16,16), C1(16,16),C2(16,16), PI(16),VL(16)
       REAL*8      VR (16)
       DIMENSION NTEXP(16,16)
       DIMENSION INDIC(16)
       COMPLEX*16 PD(16,16),PV(16,16),DET,DIAG(16)
       REAL*8     B(16,16)
       COMPLEX*16CI(16,16),CCR(16,16),CVL(16),CVR(16),DV(16),DUMMY(16,16)
       REAL*8 TITLE(12)
       REAL*4 LABEL
       DIMENSION X(200), Y(200)
       DATA LABEL/'    '/
       FORMAC OPTIONS
       NM=16
       TS=56.000
       EMPTY=0.00
       READ (5,500) N,K,NP,NPRINT
       WRITE(6,500) N,K,NP,NPRINT
       P=DFLOAT(NP)
       DO 100 I = 1, N
```

```
      READ (5,500)(INDIC(J),J=1,N)
      WRITE(6,500)(INDIC(J),J=1,N)
      DO 103 J=1,N
  103  T(I,J) = DFLOAT(INDIC(J))
      WRITE(6,511)(T(I,J),J=1,N)
  511 FORMAT(8D16.5)
      READ (5,500) ( NTEXP (I,J),J=1,N)
      WRITE(6,500) ( NTEXP (I,J),J=1,N)
  100 CONTINUE
      DO 110 I=1, N
      READ(5,500)(INDIC(J),J=1,N)
      WRITE(6,500)(INDIC(J),J=1,N)
      DO 104 J=1,N
  104 E(I,J)=DFLOAT(INDIC(J))
      WRITE(6,511)(E(I,J),J=1,N)
  110 CONTINUE
      READ(5,500) NPTS,MC,ITYPE,IX,IY,MDX,MDY,IW,IH,IG
      WRITE(6,500) NPTS,MC,ITYPE,IX,IY,MDX,MDY,IW,IH,IG
      READ (5,501) XSC,YSC
      WRITE(6,501) XSC,YSC
      READ (5,502)(TITLE(I),I=1,12)
      WRITE(5,502)(TITLE(I),I=1,12)
  502 FORMAT(6A8)
  500 FORMAT (16I5)
  501 FORMAT (16F5.0)
      DO 120 I=1, N
      DO 120 J=1, N
      NPWR=NTEXP(I,J)+1
      C1 (I,J) = T(I,J)* E(I,J)/ P**NPWR
      C2 (I,J) = C1(I,J)*(1.D0+E(I,J))/P
      WRITE(6,610) I,J,C1(I,J) ,I,J,C2(I,J)
  610 FORMAT('     C1(',I2,',',I2,')=',D12.7, '     C(',I2,',',I2,')=',
     2        D12.7)
  120 CONTINUE
      DO 130 I=1,N
      DO 130 J=1,N
      NPWR=NTEXP(J,I)
      B(I,J)= T(J,I) /P**NPWR
      T(J,I)=B(I,J)
      WRITE(6,628)J,I,T(J,I),EMPTY
      A(J,I)=B(I,J)
      IF (I.EQ.J) B(I,J)= B(I,J)-1.
  130 CONTINUE
      DO 140 J=1, N
      PI (J)= 0.
      B( N,J) = 1.
  140 CONTINUE
      PI (N) = 1.
```

```
      CALL DLINE ( 8, PI, N, NPRINT )
      WRITE ( 6,601) ( PI(I), I=1,N)
  601 FORMAT (/ ' STATIONARY  VECTOR', / 8D16.7,/8D16.7 // )
      V=0.D0
      SUM=0.D0
      DO 150 J=1,N
      VL(J)=0.D0
      VR(J)=0.D0
      DO 150 I=1,N
      VL(J)=VL(J)+PI(I)*C1(I,J)
      VR(J)=VR(J)+C1(J,I)
      V = V + PI(I)* C2(I,J)
      SUM=SUM+PI(I)*C1(I,J)
  150 CONTINUE
      WRITE(6,609)
  609 FORMAT('     LEFT AND RIGHT VECTORS')
      WRITE(6,608)      (VL(I),I=1,N)
      WRITE(6,608)      (VR(I),I=1,N)
  608 FORMAT( 8D16.7)
      WRITE(6,611)SUM
  611 FORMAT('      EXPECTATION=',D14.7)
      VAR=V-SUM*SUM
      WRITE(6,612)   VAR
  612 FORMAT('        VARIANCE=',D14.7)
      CALL DEIGNP(N,NM,A,TS,EVR,EVI,VECR,VECI,INDIC)
      DO 155 I=1,N
      CVL(I)=DCMPLX(VL(I),EMPTY)
      CVR(I)=DCMPLX(VR(I),EMPTY )
      WRITE(6,602) I, INDIC (I)
  602 FORMAT(///'      INDICATOR(',I2,')=',I1)
       IF (INDIC(I).EQ.0) GO TO 155
      WRITE(6,603) I,EVR(I),EVI(I)
  603 FORMAT(/'       LAMBDA(',I2,'=',D14.7,'+ I*',D14.7)
      IF (INDIC(I).EQ.1) GO TO 155
      WRITE (6,604) I
  604 FORMAT (/'       EIGENVECTOR FOR LAMBDA(',I2,'): ')
      WRITE(6,605) (VECR(J,I),VECI(J,I),J=1,N)
  605 FORMAT(16X,D16.7,' +  I*',D14.7)
  155 CONTINUE
      DO 157 I=1,N
      R=EVR(I)
      C=EVI(I)
      DIAG(I)=DCMPLX(R,C)
      DO 157 J=1,N
      R=VECR(I,J)
      C=VECI(I,J)
      PD(I,J)=DCMPLX(R,C)
      PV(I,J)=PD(I,J)
```

```
      CI(I,J)=(0.D0,0.D0)
      IF(I.EQ.J)CI(J,J)= (1.D0,0.D0)
  157 CONTINUE
      CALL DCMXIN(N,PV,NM,DET)
      DO 156 I=1,N
      WRITE(6,606) I
  606 FORMAT(//'     P INVERSE  ',I2,'-TH ROW')
      DO 156 J=1,N
      R=DREAL(PV(I,J))
      C=DIMAG(PV(I,J))
      WRITE(6,605)R,C
  156 CONTINUE
      NTRY =0
  153 CONTINUE
      DIF=0.D0
      DO 151 I=1,N
      DO 151 J=1,N
      DET=(0.D0,0.D0)
      DO  152 L=1,N
        DET = DET + PD(I,L)*PV(L,J)
  152 CONTINUE
      R =DREAL (DET)
      C = DIMAG(DET)
      WRITE (6,626)I,J,R,C
  626 FORMAT('    IDENT(',I2,',',I2,')=',D14.7,' +I*',D14.7)
  627 FORMAT('    INVRS(',I2,',',I2,')=',D14.7,' +I*',D14.7)
  151 CONTINUE
      DO 225 I=1,N
      DO 225J=1,N
      DET=(0.D0,0.D0)
      DO 220 M=1,N
      DO 220 L=1,N
      R=T(L,M)
      COMP=DCMPLX(R,0.D0)
      DET=DET+PV(I,L)*COMP*PD(M,J)
  220 CONTINUE
      R=DREAL(DET)
      C=DIMAG(DET)
      WRITE(6,625)I,J,R,C
  625 FORMAT('    DIAG(',I2,',',I2,')=',D14.7, '+I*',D14.7)
  225 CONTINUE
      DO 250 I=1,N
      DO 250 J=1,N
      DET=(0.D0,0.D0)
      DO 240 K=1,N
      DET=DET+PD(I,K)*DIAG(K)*PV(K,J)
  240 CONTINUE
      R=DREAL(DET)
```

```
      C=DIMAG(DET)
      WRITE(6,628)I,J,R,C
  628 FORMAT('          T(',I2,',',I2,')=',D14.7,'+ I*',D14.7)
  250 CONTINUE
      CALL CVLMLT(CVL,PD,DV,N,NM)
      CALL CVRMLT(CVR,PV,CVL,N,NM  )
      DO 159 I=1,N
      CVR(I)=CVL(I)
      CVL(I)=DV(I)
      VRR=DREAL(CVR(I))
      VRI=DIMAG(CVR(I))
      VLR=DREAL(CVL(I))
      VLI=DIMAG(CVL(I))
  607 FORMAT ('    CVL(',I2,')=',D14.7,'+I*',D14.7,'     CVR(',I2,
     2   ')=',D14.7,'+I*',D14.7)
      WRITE(6,607)I,VLR,VLI,I,VRR,VRI
  159 CONTINUE
      DO 158 I=1,N
      DV(I)=CVL(I)*CVR(I)
  158 CONTINUE
      DO 160 I=1,N
      LET(I='I'       )
      R=EVR(I)
      C=EVI(I)
      LET(R='R')
      LET(C='C')
      LET(U=2*(X-R-#I*C))
      LET(DM(I)=1-2*R*X-2*C*X*#I+R**2+2*R*C*#I-C**2)
      R=DREAL(DV(I))
      C=DIMAG(DV(I))
      LET(R='R')
      LET(C='C')
      LET(      NM(I)=EXPAND(U*(R+C*#I)))
  160 CONTINUE
      LET(SUM1=1)
      LET (SUM3=0)
      DO 170 I=1,N
      LET(I='I')
      LET(U=SUM1*DM(I))
      LET(SUM1=EXPAND(U))
      LET(SUM2=1)
      DO 165 J=1,N
      LET(J='J')
      IF(I.EQ.J)GO TO 165
      LET(U=SUM2*DM(J))
      LET(SUM2=EXPAND(U))
  165 CONTINUE
      LET(U=SUM2*NM(I))
```

```
      LET(V=SUM3+EXPAND(U))
      LET(SUM3=V)
  170 CONTINUE
      LET(VAR='VAR')
      LET(V='V')
      LET(NMR=EXPAND(V*SUM1)+SUM3)
      LET(DNM=EXPAND(VAR*SUM1*3.141592))
      PRINT OUT (NMR)
      PRINT OUT (DNM)
      DELTA=3.141592/FLOAT(NPTS-1)
      DO 175 I=1,NPTS
      OMEGA=DELTA*DFLOAT(I-1)
      X(I)=OMEGA
      U=DCOS(OMEGA)
      LET(U='U')
      LET(FNMR=REPLACE(NMR,X,U)  )
      LET(FDNM=REPLACE(DNM,X,U))
      LET(FZERO=FNMR/FDNM)
      Y(I)=ARITH(FZERO)
  175 CONTINUE
      WRITE(6,629)(Y(I),I=1,NPTS)
      Y(1)=0.
      CALL DRAW(NPTS,X,Y,MC,ITYPE,LABEL,TITLE,XSC,YSC,IX,IY,MDX,MDY,
     2          IW,IH,IG,LAST)
      WRITE(6,630) LAST
  629 FORMAT(10E12.4)
  630 FORMAT(' LAST=',I2)
      STOP
      END


      SUBROUTINE CMULT(N,A,B,C,ND)
CMULT FORMS THE PRODUCT OF TWO NXN COMPLEX MATRICES
      COMPLEX*16 A(ND,1),B(ND,1),C(ND,1)
      DO 10 I=1,N
      DO 10 J=1,N
      DO 10 K=1,N
      C(I,J)=A(I,K)*B(K,J) + C(I,J)
   10 CONTINUE
      RETURN
      END


      SUBROUTINE CVLMLT(A,B,C,N,ND)
CVLMLT FORMS THE LEFT HAND PRODUCT OF THE VECTOR A WITH THE MATRIX B
      COMPLEX*16 A(16),B(16,16),C(16)
      DO 10 I=1,N
      C(I)=(0.D0,0.D0)
```

```
          (SEMI-MARKOV REPRESENTATION)
C   PROGRAM TO ESTIMATE THE SPECTRUM OF A SEMI-MARKOV PROCESS BY  THE
C   METHOD OF PARTIAL SUMS

    INPUT PARAMETERS
       N       THE NUMBER OF POINTS IN THE SPECTRUM APPROXIMATION
       K       THE SHAPE FACTOR OF THE ERLANG COMPONENT PROCESS
       NP      THE NUMBER OF COMPONENTS   (NP=P)
       NPRINT  LINEAR EQUATIONS PARAMETER
       NPR     2**NPR+1 CORRELATIONS WILL BE COMPUTED
       NAR     THE AUTOREGRESSIVE ORDER OF THE SIMILAR ARMA PROCESS
       NMA     THE MOVING AVERAGE ORDER OF THE SIMILAR  ARMA PROCESS
       NBSPC   BISPECTRUM COMMAND   NBSPC=0 - NO BISPECTRUM
                                    NBSPC=1 - BISPECTRUM ONLY
                                    NBSPC=-1 - SPECTRUM AND BISPECTRUM
                                               CORRELATION DAMPING NOT DONE
       NPLOT, MC, ITYPE, IX, IY, MDX, MDY, IWIDE, IHIGH, IGRID    -
                                    PLOT PARAMETERS
       ZXSC, ZYSC   PLOT PARAMETERS
       TITLE   PLOT PARAMETER
       T       TRANSITION MATRIX(NUMERATORS)
       NTEXP   TRANSITION MATRIX (DENOMINATOR COEFFICIENTS)
       E       SHAPE PARAMETER MATRIX

    SUBROUTINES CALLED
       DLINE - SOLVES SYSTEM OF LINEAR EQUATIONS DOUBLE PRECISION
       DRAW - PLOT ROUTINE
       BISPEC - COMPUTES THE BISPECTRUM - INCLUDED IN THIS APPENDIX

       MULT, VMULT    INCLUDED WITH THIS LISTING

       IMPLICIT REAL*8 (A-H,O-Y)
       DIMENSION Z(512), ZX(512),TITLE(12)
       DIMENSION T(32,16,16),E(16,16),C1(16,16),PI(16),VL(16),VR(16)
       DIMENSION C2(16,16),W(16,16), NTEXP(16,16)   ,TEST(16,16),ALFA(16)
       DIMENSION C3(16,16)
 1000  CONTINUE
       READ(5,1,END=2000) N,K,NP,NPRINT , NPR ,NAR,NMA,NBSPC
       WRITE(6,1)N,K,NP,NPRINT, NPR,NAR,NMA,NBSPC
       READ(5,1) NPLOT,MC,ITYPE,IX,IY,MDX,MDY,IWIDE,IHIGH,IGRID
       WRITE(6,1)NPLOT,MC,ITYPE,IX,IY,MDX,MDY,IWIDE,IHIGH,IGRID
       READ(5,2) ZXSC, ZYSC
       WRITE(6,2)ZXSC,ZYSC
     1 FORMAT (16I5)
```

```
      IF (MC.LE.1 ) READ(5,8)(TITLE(I),I=1,12)
      IF(MC.LE.1)WRITE(6,8)(TITLE(I),I=1,12)
    8 FORMAT (6A8)
      DO 10  I=1,N
      READ (5,2) ( T(1,I,J),J=1,N)
      WRITE (6,2)(T(1,I,J),J=1,N)
    2 FORMAT(16F5.0)
      READ (5,1) ( NTEXP(I,J),  J=1,N)
      WRITE(6,1)(NTEXP(I,J),J=1,N)
   10 CONTINUE
      DO 20  I=1,N
      READ (5,2) (E(I,J),J=1,N )
      WRITE(6,2)(E(I,J),J=1,N)
   20 CONTINUE
      DATA  ZLABEL /'     '/, PII/ 3.141592653589793/
      P=DFLOAT(NP)
      NPWR=4
      DO 30 I=1,N
      DO 30 J=1,N
      NPWR=NTEXP(J,I)
      W(I,J)= T(1,J,I)/P**NPWR
      TEST(I,J)=W(I,J)
   30 CONTINUE
      DO 40 I=1,N
      W(I,I)= W(I,I)-1.D0
      TEST(I,I)=W(I,I)
      W(N,I)=1.D0
      TEST(N,I)=1.D0
      PI(I)=0.D0
   40 CONTINUE
      PI(N)=1.D0
      CALL DLINE (W,PI,N,NPRINT)
   45 CONTINUE
      DO 51 I=1,N
      DD=PI(I)
  600 FORMAT ('      PI(',I2,')=',D32.14)
  601 FORMAT('      TEST(',I2,',',I2,')=',D32.14,'       SUM=',D32.14)
      WRITE(6,600)I,DD
   51 CONTINUE
      DO 60 I=1,N
      DO 60 J=1,N
      NXP= NTEXP(I,J)
      T(1,I,J)= T(1,I,J)/P**NXP
      C1(I,J)=T(1,I,J)*E(I,J)/P
      C2(I,J)=C1(I,J)*(1.+E(I,J))/P
      C3(I,J)=C2(I,J)*(2.D0+E(I,J))/P
   60 CONTINUE
      M=1
```

```
      WRITE(6,603)M
      DO 61 K=1,N
      WRITE(6,602)(T (1,K,L),L=1,N)
      SUM=0.D0
      DO 63 L=1,N
      SUM=SUM+T(1,K,L)
   63 CONTINUE
      WRITE(6,613) K,SUM
  613 FORMAT('    ROW',I3,' ADDS TO ', D24.12)
   61 CONTINUE
      DO 65 I=1,NPR
      JTOP=2**(I-1)
      DO 65J=1,JTOP
      CALL MULT(T,JTOP,J,N)
      M=JTOP+J
      WRITE(6,603) M
  603 FORMAT('       T(' ,I2,')')
      DO 65 K=1,N
      WRITE(6,602)(T (M,K,L),L=1,N)
  602  FORMAT (5D24.12)
   65 CONTINUE
      DO 70 I=1,N
      VL(I)=0.D0
      VR(I)=0.D0
      DO 70 J=1,N
      VL(I)=VL(I)+ PI(J)*C1(J,I)
      VR(I)=VR(I)+ C1(I,J)
   70 CONTINUE
      NR=2**NPR+1
      IF ( NBSPC.LE.0) GO TO 75
  700 CONTINUE
      DO 71 J=1,N
      ABLE=PI(J)
      DO 71 I=1,N
      TEST(I,J)=ABLE
   71 CONTINUE
      CALL BISPEC(T,C1,C2,C3,VL,VR,NPLOT,NR,TEST,PI,N)
      WRITE(6,72)
   72 FORMAT('    BISPECTRUM PLOT COMPLETE')
      STOP
   75 CONTINUE
      JTOP=2*JTOP
      DO 80 I=1,JTOP
      CALL VMULT (VL,T,VR,ALFA,I,N)
   80 CONTINUE
      SUM=0.D0
      V=0.D0
      ALPHA=0.D0
```

```
      DO 90 I=1,N
      ALPHA=ALPHA+VL(I)*VR(I)
      DO 90 J=1,N
      SUM=SUM+ PI(J)*C1(J,I)
      V=V+PI(J)*C2(J,I)
   90 CONTINUE
      WRITE(6,604) SUM
  604 FORMAT('      EXPECTATION=',D22.14)
      SUM=SUM*SUM
      V=V-SUM
      WRITE (6,5) V
    5 FORMAT ( '  VARIANCE =',D24.14)
      DO 100 I=1,JTOP
      ALFA(I)= ALFA(I)-SUM
  100 CONTINUE
      ALPHA=ALPHA-SUM
      NR=JTOP+1
      WRITE(6,3) NR
      WRITE(6,610) ALPHA,(ALFA(I),I=1,JTOP)
    3 FORMAT('   FIRST ',I2,' SERIAL COVARIANCES')
  610 FORMAT (4D32.14)
      DO 110 I=1,16
      ALFA(I)=ALFA(I)/V
  110 CONTINUE
      ALPHA = ALPHA/V
      WRITE(6,4)NR
      WRITE(6,610) ALPHA,(ALFA(I),I=1,JTOP)
    4 FORMAT (//'    FIRST ',I2,' SERIAL CORELATIONS')
      DELTA=PII/ DFLOAT(NPLOT-1)
      DO 120 I=1,NPLOT
      THETA=DELTA*DFLOAT(I-1)
      OMEGA=1.D0
      Z(I)=1.D0 +DCOS(THETA)*2.D0*ALPHA
      ZX(I)=THETA
      DO115 J=1,JTOP
      OMEGA=THETA*DFLOAT(J+1)
      Z(I)=DCOS(OMEGA)*2.D0*ALFA(J)+Z(I)
  115 CONTINUE
      Z(I)=Z(I)/PII
  120 CONTINUE
      WRITE (6,6)
      WRITE (6,7)(Z(I),I=1,NPLOT)
    6 FORMAT ( /'  SPECTRAL DENSITY' )
    7 FORMAT ( 8E16.7 )
      WRITE(6,1)NPLOT,MC,ITYPE,IX,IY,MDX,MDY,IWIDE,IHIGH,IGRID
      CALL DRAW (NPLOT, ZX, Z, MC, ITYPE, ZLABEL, TITLE, ZXSC, ZYSC,
     2           IX, IY, MDX, MDY, IWIDE, IHIGH, IGRID, LAST )
      WRITE (6,9) LAST
```

```
    9 FORMAT (  ' LAST=',I2,'1')
      IF (NBSPC.EQ.-2) GO TO 700
      IF( NAR.LT.0.OR.NAR.GT.10)GO TO 1000
      J=NMA-NAR
      IF( NAR.EQ.1) GO TO 250
      DO 210 I=1,NAR
      DO 220 L=1,NAR
  220 E(I,L)= ALFA(J+I+L-2)
  210 PI(I) = ALFA(J+I+NAR-1)
      CALL DLINE (E,PI,NAR,NPRINT)
  240 CONTINUE
      WRITE (6,611)
      WRITE (6,610) ( PI(I),I=1, NAR)
  611 FORMAT ( 'DAMPING FACTORS' )
      TEST1=0.D0
      TEST2=0.D0
      DO 230 I=1, NAR
      TEST1=TEST1+PI(I)*ALFA(J+1+I)
      TEST2=TEST2+PI(I)*ALFA(J+2+I)
  230 CONTINUE
      L=J+2+NAR
      DIF= TEST1-ALFA(L)
      M=L+1
      WRITE(6,612) TEST1,M,ALFA(L),DIF
      L=M
      M=L+1
      DIF=TEST2-ALFA(L)
      WRITE(6,612)   TEST2,M,ALFA(L),DIF
  612 FORMAT( ' TEST=',D20.13,'   RHO(',I2,')=',D20.13,'   DIF=',D20.13)
      IF (NBSPC.EQ.-1) GO TO 700
      GO TO 1000
  250 CONTINUE
      PI(1)= ALFA(J+1)/ALFA(J)
      GO TO 240
 2000 STOP
      END


      SUBROUTINE MULT (X,I1,I2,N)
      REAL*8 X(32,16,16),SUM
      I3=I1+I2
      DO 10 I=1,N
      DO 10 J=1,N
      SUM=0.D0
      DO  5 K=1,N
      SUM=SUM + X(I1,I,K)* X(I2,K,J)
    5 CONTINUE
      X(I3,I,J)=SUM
```

```
 10 CONTINUE
    RETURN
    END


    SUBROUTINE VMULT (X,A,Y,ALFA,I1,N)
    REAL*8 X(16),A(32,16,16),Y(16),ALFA(16),SUM
    SUM=0.D0
    DO 10 I=1,N
    DO 10 J=1,N
    SUM= SUM + X(I)*A(I1,I,J)*Y(J)
 10 CONTINUE
    ALFA (I1)=SUM
    RETURN
    END
```

```
      (SEMI-MARKOV REPRESENTATION)
   BISPEC COMPUTES AND PLOTS THE BISPECTRUM OF AN E(K,P) PROCESS
   USING THE TRANSITION MATRIX, THE MOMENT MATRICES AND
   THE STATIONARY VECTOR
 THIS PROGRAM IS DESIGNED TO BE CALLED BY THE E(K,P) APPROXIMATION ROUTINE

CALLING PARAMETERS
   T       TRANSITION MATRIX((16,16)
   T       TRANSITION MATRIX (16,16)
   C1      MATRIX OF FIRST MOMENTS (16,16)
   C2      MATRIX OF SECOND MOMENTS
   C2      MATRIX OF SECOND MOMENTS (16X16)
   C3      MATRIX OF THIRD MOMENTS (16X16)
   VL      PRODUCT OF STATIONARY VECTOR WITH C1 (16)
   VR      PRODUCT OF C1 WITH THE 1 VECTOR (16)
   NPTS    THE NUMBER OF GRID POINTS SAMPLED EACH AXIS  NPTS<41
   N       THE NUMBER OF POWERS OF T PROVIDED
   TINF    THE STATIONARY TRANSITION MATRIX
   PI      THE STATIONARY VECTOR
   NDIM    THE DIMENSION OF THE STATE SPACE

SUBROUTINES CALLED
   PLT3D1    PLOT ROUTINE
   LVMLT, VVMLT, MULTTL, MULTR, SMULT, MSUB, MADD, MULTTR,
                    INCLUDED WITH THIS LISTING

   SUBROUTINE BISPEC ( T, C1, C2, C3, VL, VR, NPTS, N, TINF, PI,NDIM)
   IMPLICIT REAL*8 (A-H,O-Y)
   LOGICAL*1 IDN(40,40)
   DIMENSION KX(103),KY(103)
   DIMENSION TTL(12)
   DIMENSION T(32,16,16), C1(16,16), C2(16,16), C3(16,16), VL(16),
  2    VR(16), TRV(33,33), W1(16,16), W2(16,16),W3(16,16), VW(16)
  3  , TINF (16,16), W4(16,16), PI(16),X(40 ),Y(40 ),ZX(40 ),
  4    ZY(40),Z(40,40)
   DIMENSION W5(16,16),TRV0(17)
   DIMENSION ZF(2), ZSZ(2)
   DIMENSION ZWK(40,40,3)
   DIMENSION TRVJ(17)
   DIMENSION            ONE(16)
   DATA ONE/16*1.D0/
   IF (NPTS.GT.40) NPTS=40
   SUMT=0.D0
   SUMB=0.D0
```

```
      CALL LVMLT(PI,C1,VW,NDIM)
      CALL VVMLT(VW,ONE,EX,NDIM)
      EX3=2.D0*EX**3
      DO 20 J=1,N
      IF ( J.EQ. 1) GO TO 1
      JC= J-1
      CALL MULTTL(T,C1,W1,JC,NDIM)
      CALL MULTTL(T,C2,W2,JC,NDIM)
    5 CONTINUE
      CALL MULTR(C1,W2,W3,NDIM)
      CALL MULTR(C1,W1,W4,NDIM)
      S=-2.D0*EX
      CALL SMULT(S,W4,W2,NDIM)
      CALL SMULT(EX,C2,W5,NDIM)
      CALL MSUB(W2,W5,W5,NDIM)
      CALL MADD(W3,W5,W3,NDIM)
      CALL LVMLT(PI,W3,VW,NDIM)
      CALL VVMLT(VW,ONE,S,NDIM)
      TRVJ(J)=S+EX3
      SUMT=SUMT+3.D0*TRVJ(J)
      WRITE(6,601)J,J,TRVJ(J)
  601 FORMAT('    TRIVARIANCE(',I2,',',I2,')=',D12.5)
      GO TO 2
    1 CONTINUE
      DO 3 IT=1,NDIM
      DO 3 JT=1,NDIM
      W1(IT,JT)=C1(IT,JT)
      W2(IT,JT)=C2(IT,JT)
    3 CONTINUE
      GO TO 5
    2 CONTINUE
      CALL MULTR(C2,W1,W3,NDIM)
      CALL MADD(W3,W5,W3,NDIM)
      CALL LVMLT(PI,W3,VW,NDIM)
      CALL VVMLT(VW,ONE,S,NDIM)
      TRVO(J)=S+EX3
      SUMT=SUMT+3.D0*TRVO(J)
      JJ=0
      WRITE(6,601)JJ,J,TRVO(J)
      DO 10 I=1,N
      IF (I.EQ.1) GO TO 7
      IC= I-1
      CALL MULTTR(W1,T,W2,IC,NDIM)
   18 CONTINUE
      IF(I.EQ.1 .OR. J.EQ.1)GO TO 9
      IF (I+J.GT.N)GO TO 21
      IJC= I+J-1
      DO 8 IT=1,NDIM
```

```
      DO 8 JT=1,NDIM
      W4(IT,JT)= T(IJC,IT,JT)+T(IC,IT,JT)+T(JC,IT,JT)
    8 CONTINUE
      GO TO 23
    9 CONTINUE
      IF (I.EQ.1 .AND. J.EQ.1) GO TO  11
      IJC=I+J-1
      IF (IJC.GE.N) IJC=N-1
      JM=MAXO(I,J)-1
      DO 12 IT=1,NDIM
      DO 12 JT=1,NDIM
      W4(IT,JT)=T(IJC,IT,JT)+T(JM,IT,JT)
      IF (IT.EQ.JT)       W4(IT,JT)=W4(IT,JT)+1.DO
   12 CONTINUE
      GO TO 23
   11 CONTINUE
      DO 13 IT=1,NDIM
      DO 13 JT=1,NDIM
      W4(IT,JT)=T(2,IT,JT)
      IF (IT.EQ.JT)W4(IT,IT)=W4(IT,IT)+2.DO
   13 CONTINUE
   23 CONTINUE
      S=-EX
      CALL SMULT(S,W4,W4,NDIM)
      CALL MADD(W2,W4,W4,NDIM)
      CALL LVMLT ( VL,W4,Vw,NDIM)
      CALL VVMLT ( Vw,VR,S,NDIM )
      TRV(J,I)=S+EX3
      SUMT=SUMT+6.DO*TRV(J,I)
      JI=J+I
      WRITE(6,601)J,JI,TRV(J, I)
      GO TO 10
   21 CONTINUE
      DO 22 IT=1,NDIM
      DO 22 JT=1,NDIM
      W4(IT,JT)=TINF(IT,JT)+T(IC,IT,JT)+T(JC,IT,JT)
   22 CONTINUE
      GO TO 23
    7 CONTINUE
      DO 16 IT=1,NDIM
      DO 16 JT=1,NDIM
      W2(IT,JT)=W1(IT,JT)
   16 CONTINUE
      GO TO 18
   10 CONTINUE
   20 CONTINUE
      S=3.DO*EX
      CALL SMULT(S,C2,W4,NDIM)
```

```
      CALL MSUB (C3,W4,W4,NDIM)
      CALL LVMLT(PI,W4,VW,NDIM)
      CALL VVMLT(VW,ONE,S,NDIM)
      TRVZ=S+EX3
      SUMT=SUMT+TRVZ
      JI=0
      WRITE(6,601)JI,JI,TRVZ
      WRITE(6,602)SUMT
  602 FORMAT('    TRIVARIANCE SUM=',D12.5)
      SUMT=SUMT/TRVZ
      SUMT=SUMT/5.289868133696
      WRITE(6,604)SUMT
  604 FORMAT('    NORMALIZED TRIVARIANCE SUM=',D14.7)
      DELTA=3.1415926536/ DFLOAT(NPTS-1)
      DO 30 L=1, NPTS
      A1= DFLOAT(L-1)*DELTA
      X (L)= A1
      Y(L)=A1
      ZX(L)=A1
      ZY(L)=A1
   30 CONTINUE
      SUM=0.D0
      DO 80 L=1,NPTS
      A1=X(L)
      DO 80 M=1,L
      A2=Y(M)
        SUMR=0.D0
      SUMI=0.D0
      DO 70 J=1,N
      G=DFLOAT (J)
      ALFAJ=(A1+A2)*G
      BETAJ=A2*G
      GAMMAJ=A1*G
      SUMR=SUMR+(TRV0(J)+TRVJ(J))*(DCOS(ALFAJ)+DCOS(BETAJ)+DCOS(GAMMAJ))
      SUMI=SUMI+(TRV0(J)-TRVJ(J))*(DSIN(ALFAJ)-DSIN(BETAJ)-DSIN(GAMMAJ))
      DO 70 I=1,N
      F= DFLOAT(I)
      ALFAI=(A1+A2)*F
      BETAI=A2*F
      GAMMAI=A1*F
      B=DCOS(ALFAJ+BETAI)+DCOS(ALFAJ+GAMMAI)+DCOS(ALFAI+BETAJ)
     2  +DCOS(ALFAI+GAMMAJ)+DCOS(BETAJ-GAMMAI)+DCOS(GAMMAJ-BETAI)
      SUMR=SUMR+TRV(J,I)*B
      B=-DSIN(ALFAJ+BETAI)-DSIN(ALFAJ+GAMMAI)+DSIN(ALFAI+BETAJ)
     2  +DSIN(ALFAI+GAMMAJ)+DSIN(BETAJ-GAMMAI)+DSIN(GAMMAJ-BETAI)
      SUMI=SUMI+TRV(J,I)*B
   70 CONTINUE
      R=(SUMR/TRVZ+1.D0)/39.478417604352
```

```
      C=(SUMI/TRVZ)/     39.478417604352
      A=R**2+C**2
      Z(L,M)=DSQRT(A)
      Z(M,L)=Z(L,M)
      WRITE(6,600)A1,A2,R,C,Z(L,M)
  600 FORMAT(' B(',F5.4,',',F6.4,')=',D14.7,'+I*',D14.7,'  ABS VAL=',
     2  D14.7)
      WRITE(7,701) ZX(L),ZY(M),Z(L,M)
  701 FORMAT (4E20.7)
      IF(L.EQ.NPTS)GO TO 80
      D=R
      IF (M.EQ.L)D=R/2.DO
      SUMB=SUMB+D
      IF(M+L.GT.NPTS)GO TO 80
      IF(M+L.EQ.NPTS)D=D/2.DO
      SUM=SUM+D
   80 CONTINUE
      SUMB=SUMB*DELTA**2
      SUM=SUM*DELTA**2
      SUMB=SUM*8.DO+SUMB*4.DO
      WRITE(6,603)SUMB
  603 FORMAT('    8ISPECTRUM SUM=',D12.5)
      READ (5,502)(TTL(I),I=1,12)
      WRITE(6,502)(TTL(I),I=1,12)
      READ(5,501)ZALFA,ZBETA,ZF(1),ZF(2),ZSZ(1),ZSZ(2)
      WRITE(6,501)ZALFA,ZBETA,ZF(1),ZF(2),ZSZ(1),ZSZ(2)
      READ(5,500) NKXY,LINES
      WRITE(6,500) NKXY,LINES
      CALL PLT3D1(ZX,NPTS,ZY,NPTS,Z,ZALFA,ZBETA,ZF,TTL,ZSZ,ZWK,IDN,
     2       KX,KY,NKXY,LINES)
  500 FORMAT(16I5)
  501 FORMAT(8F10.5)
  502 FORMAT(6A8)
      RETURN
      END


      SUBROUTINE  VVMLT (A,B,C, N  )
      REAL*8 A(16), B(16), C
      C=0.DO
      DO 10 I=1,N
      C= C+ A(I)*B(I)
   10 CONTINUE
      RETURN
      END


      SUBROUTINE RVMLT (A,B,C,N)
```

```
      REAL*8 A(16,16),B(16),C(16),SUM
      DO 10 I=1,N
      SUM=0.DO
      DO 5 J=1, N
      SUM= SUM+ A(I,J)*B(J)
    5 CONTINUE
      C(I)= SUM
   10 CONTINUE
      RETURN
      END

      SUBROUTINE LVMLT (A,B,C,N)
      REAL*8 A(16), B(16,16), C(16), SUM
      DO 10 I=1,N
      SUM=0.DO
      DO 5 J=1, N
      SUM=SUM + A(J)* B(J,I)
    5 CONTINUE
      C(I)= SUM
   10 CONTINUE
      RETURN
      END
      SUBROUTINE MADDT (A,B,C,IK,N )
      REAL*8 A(32,16,16), B(16,16), C(16,16)
      DO 10 I=1,N
      DO 10 J=1,N
      C(I,J)= A(IK,I,J)+ B(I,J)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MSUB (A,B,C, N)
      REAL*8 A(16,16),B(16,16),C(16,16)
      DO 10 I=1,N
      DO 10 J=1,N
      C(I,J)= A(I,J)-B(I,J)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MADD (A,B,C,N)
      REAL*8 A(16,16),B(16,16), C(16,16)
      DO 10 I= 1,N
      DO 10 J=1, N
      C(I,J)= A(I,J)+B(I,J)
```

```
10 CONTINUE
   RETURN
   END

   SUBROUTINE MULTTL (A,B,C,IK,N)
   REAL*8 A(32,16,16),B(16,16),C(16,16),SUM
   DO 10 I=1,N
   DO 10 J=1,N
   SUM=0.DO
   DO 5 K=1,N
   SUM= SUM+ A(IK,I,K)*B(K,J)
 5 CONTINUE
   C(I,J)=SUM
10 CONTINUE
   RETURN
   END

   SUBROUTINE MULTR (A,B,C,N)
   REAL*8 A(16,16), B(16,16), C(16,16),SUM
   DO 10 I=1,N
   DO 10 J=1,N
   SUM=0.DO
   DO 5  K=1,N
   SUM= SUM+ A(I,K)*B(K,J)
 5 CONTINUE
   C(I,J)=SUM
10 CONTINUE
   RETURN
   END

   SUBROUTINE MSUBT(A,B,C,IK,N)
   REAL*8 A(32,16,16),B(16,16) ,C(16,16)
   DO 10 I=1,N
   DO 10 J=1,N
   C(I,J)=A(IK,I,J)-B(I,J)
10 CONTINUE
   RETURN
   END

   SUBROUTINE MULTTR(A,B,C,IK,N)
   REAL*8 A(16,16),B(32,16,16),C(16,16)
   DO 10 I=1,N
   DO 10 J=1,N
   SUM=0.DO
```

```
      DO 5 K=1,N
      SUM=SUM+A(I,K)*B(IK,K,J)
    5 CONTINUE
      C(I,J)=SUM
   10 CONTINUE
      RETURN
      END

      SUBROUTINE SMULT(A,B,C,N)
      REAL*8 A, B(16,16),C(16,16)
      DO 10 I=1,N
      DO 10 J=1,N
      C(I,J)=A*B(I,J)
   10 CONTINUE
      RETURN
      END
```

```
COUNT SPECTRUM FOR ERLANG RENEWAL PROCESS NORMALIZED BY RATE=K/P

   INPUT PARAMETERS ---
      K       THE SHAPE FACTOR OF THE ERLANG COMPONENT PROCESS
      MC      NUMBER OF CURVES TO BE PLOTTED    MC=0 - ONE CURVE
                                                MC=1 - FIRST OF SEVERAL
                                                MC=2 - INTERMEDIATE OF SEVERAL
                                                MC=3 - FINAL CURVE
      ITYPE   PLOT PARAMETER
      NP      THE NUMBER OF COMPONENT PROCESSES  (NP=P)
      NWAY    METHOD OF COMPUTATION   NWAY = 0 - POLYNOMIAL IN (W*P/K)
                                      NWAY = 1 - FORMULA IN COS(W*P/K)
      NPTS    THE NUMBER OF POINTS IN THE NUMERICAL APPROXIMATION
                                      NPTS<201
      IX, IY, MDX, MDY, IW, IH, IG    PLOT PARAMETERS
      AMAX    THE UPPER LIMIT OF THE X AXIS
      XSC, YSC    PLOT PARAMETERS
      SCALE   A MULTIPLIER FOR G  FOR PLOT PURPOSES
      TITLE   PLOT PARAMETER

      DIMENSION F1(10),F2(5),X(200),G(200)
      REAL*8 TITLE(12)
      DIMENSION WORDK(8)
      DATA WORDK/' K=1',' K=2',' K=3',' 4,5','     ',' K=6',' K=7',' K=8'
     2     /
   10 CONTINUE
      READ (5,500,END=1000) K,MC, ITYPE,NP,NWAY
      WORD=WORDK(K)
      WRITE(6,500,END=1000) K,MC, ITYPE,NP,NWAY
      IF ( MC.LE.1) READ (5,500 ) NPTS,     IX,IY,MDX,MDY,IW,IH,IG
      IF ( MC.LE.1) WRITE(6,500 ) NPTS,     IX,IY,MDX,MDY,IW,IH,IG
      IF (MC.LE.1) READ (5,501) AMAX,XSC,YSC,SCALE
      IF (MC.LE.1) WRITE(6,501) AMAX,XSC,YSC,SCALE
      IF (MC.LE.1)  READ(5,502)(TITLE(I),I=1,12)
      IF (MC.LE.1) WRITE(6,502)(TITLE(I),I=1,12)
  500 FORMAT (16I5)
  501 FORMAT (8F10.3)
  502 FORMAT (6A8)
      FK=FLOAT(K)
      FKP=FK/FLOAT(NP)
      IF(NWAY .EQ. 1) GO TO 200
  199 CONTINUE
      F1(1)=FK
      F2(1)=F1(1)*(FK-1.)   /2.
```

```
      DO 20 J=2,K
      F1(J)=F1(J-1)*FLOAT(K-J+1)/FLOAT(J)
      M=2*J
      IF(M.GT. K) GO TO 20
      F2(J)=F2(J-1)*FLOAT(K-M+1)*FLOAT(K-M+2)      /(FLOAT(M)*FLOAT(M-1))
   20 CONTINUE
      DELTA = AMAX/FLOAT(NPTS-1)
      DO 100 I=1,NPTS
      W=DELTA*FLOAT (I-1)
      X(I)=W
      W=( W*3.141592/FKP)**2
      SUM1=F1(1)
      SUM2=-F2(1)
      DO 50 J=2,K
      M=2*J
      SUM1=SUM1+F1(J)*(W**(J-1)          )
      IF(M.GT.K)GO TO 50
      SUM2=SUM2+ F2(J)*(W**(J-1))*((-1)**J          )
   50 CONTINUE
      T    = SUM1/(SUM1-2.*SUM2)
      G(I)=     T/ (3.141592)*SCALE
  100 CONTINUE
  101 CONTINUE
      WRITE(6,605)(X(I),I=1,NPTS          )
      WRITE(6,605)(G(I),I=1,NPTS          )
  605 FORMAT (10E12.4)
      CALL DRAW (NPTS,X,G,MC,ITYPE,WORD,TITLE,XSC,YSC,IX,IY,MDX,MDY,
     2           IW,IH,IG,LAST)
      WRITE (6,601) LAST
  601 FORMAT ( '    LAST=',I1)
      GO TO 10
 1000 STOP
 1111 CONTINUE
  200 CONTINUE
      DELTA=AMAX/FLOAT(NPTS-1)
      DO 250 I=1,NPTS
      W=DELTA*FLOAT(I)
      X(I)=W
      FK=FKP
      W=W*3.14159
      S=ATAN(W/FK)
      FKS=FK*S
      C=COS(S)
      CK=COS(FKS)
      CSK=C**(2*K)
      TOP=1-CSK
      DEN=1-2.*C**K*CK+CSK
      T=TOP/DEN
```

```
      G(I)=T*SCALE/3.14159
  250 CONTINUE
      GO TO 101
  251 CONTINUE
      END
```

```
C   DECK 4-10    PROGRAM TO CALCULATE THE MARGINAL INTERVAL DENSITY
C                FOR THE SUPERPOSITION OF P IID ERLANG  DISTRIBUTED
C                  RENEWAL  PROCESSES WITH SHAPE PARAMETER K

    INPUT PARAMETERS
       K       THE SHAPE PARAMETER OF THE COMPONENT PROCESSES
       NP      THE NUMBER OF COMPONENT PROCESSES
       NPLOT, NY, NT    PLOT PARAMETERS

    SUBROUTINES CALLED
       DRAW    PLOT ROUTINE
       LPLOT - PLOT ROUTINE

       REAL*8 L3(12)
       DIMENSION X(512)
       DIMENSION Y(512),NA(512),L1(30)
       DATA LABEL/'    '/
       LAST=99
       FORMAC OPTIONS
       LET ( FNC (G)=($(2)*R)**$(1)*X**($(1)-1)/FAC ($(1)-1) )
    75 CONTINUE
       READ (5,1,END=99)K,NP,NPLOT,NY,NT
       WRITE(6,17) K,NP,NPLOT,NY,NT
    17 FORMAT ( ///'  INPUT PARAMETERS ---K=',I2,'   P=',I2,'   NPLOT=',
      2 I2,'   NY=',I2,'   NT=',I2///)
       IF(K.EQ.1)GO TO 98
       IF (NP.EQ.1)GO TO 120
       LET ( K = 'K' )
       LET ( P = 'NP')
     1 FORMAT ( 1615 )
       OPTSET ( INT )
       JEND = K
       LET ( S = 0 )
       LET ( T = 0 )
       DO 10 J = 1, JEND
       LET ( J = 'J' )
       LET ( A(J) = K+1-J )
       LET ( C = R*A(J)*COMB(K+J-2,K-1)*H(K+J-1,2)/(2**(K+J-1))+S )
       LET ( S = C )
       LET ( C = 2*(P-1)*R*COMB(2*J-2,J-1)*H(2*J-1,2)/ 4**J + T )
       LET ( T = C )
       LET ( B(J) = A(J))
    10 CONTINUE
       LET ( U = 0 )
```

```
      JEND = JEND-1
      DO 20 J =1,JEND
      LET ( J='J')
      DO 20 N =J,JEND
      LET ( N='N'+1)
      LET ( C = R*(P-1)     *COMB(J+N-2,J-1)*H(J+N-1,2)/2**(J+N-2)+U )
      LET ( U = C )
   20 CONTINUE
      LET ( F = T+S+U )
      NEND = K
      JEND = K
      LET (M=1)
      NPOWER = NP-3
      LET ( NP = 'NPOWER'+2 )
      IF ( NPOWER .LT. 1 ) GO TO 51
      DO 50 I=1,NPOWER
      LET ( T=0)
      LET ( L=M+1)
      DO 30 J=1,JEND
      LET ( J='J')
      DO 30 N =1,NEND
      LET ( N ='N ')
      LET ( C = R*A(J)*B(N)*M**N*COMB (J+N-2,J-1)*H(J+N-1,L)/L**(J+N-1)
     2        + T )
      LET ( T = C )
   30 CONTINUE
      LET ( M = L )
      NEND = JEND + NEND - 1
      DO 40 N = 1, NEND
      LET ( N= 'N' )
      LET ( B(N)= COEFF (T,H(N,M)))
   40 CONTINUE
   50 CONTINUE
   51 CONTINUE
      JEND = 2*K-1
      IF (NP.GT.2) GO TO 52
      LET (S=F)
      LET (L=2)
      GO TO 61
   52 CONTINUE
      LET ( S = 0 )
      LET ( L = 2+M )
      DO 60 J = 1, JEND
      LET ( J = 'J' )
      LET ( A(J) = COEFF(F,H(J,2)))
      DO 60 N= 1, NEND
      LET ( N = 'N' )
      LET ( C = R*A(J)*B(N)*2**J*M**N *COMB(J+N-2,J-1)*H(J+N-1,L)/
```

```
  2          L**(J+N-1)  + S )
   LET ( S = C )
60 CONTINUE
   JEND = JEND + NEND - 1
61 CONTINUE
   LET ( F = R**NP*K**NP )
   LET ( SUMA = 0 )
   LET ( EX = 0 )
   LET ( EX2 = 0 )
   LET (F1=0)
   WRITE (6,2) K, NP
   DO 70 J = 1, JEND
   LET ( J = 'J' )
   LET ( A(J)= COEFF(S,H(J,L))/F )
   LET (C = A(J)*H(J,L) + F1)
   LET (F1 = C )
   LET ( C = SUMA + A(J) )
   LET ( SUMA = C )
   LET ( C = EX + J*A(J)/(P*R))
   LET ( EX = C )
   LET ( C = EX2 + J*(J+1)*A(J)/ (P**2*R**2))
   LET ( EX2 = C )
   PRINT OUT ( A (J) )
70 CONTINUE
   LET (VAR =EX2 - EX**2 )
   LET ( CV2 = VAR/ EX**2 )
   LET ( F = 0 )
   DO  80 J = 1, JEND
   LET ( J = 'J')
   LET ( C = A(J)* G(J,L) + F )
   LET ( F = C )
80 CONTINUE
   LET ( C = F* E**(-L*R*X ))
   LET ( F = C )
   PRINT OUT (F1)
   PRINT OUT ( F )
   PRINT OUT ( SUMA)
   PRINT OUT ( EX)
   PRINT OUT (EX2 )
   PRINT OUT ( VAR )
   PRINT OUT ( CV2 )
   LET ( GPLUS = P*R/(K**2*PI))
   PRINT OUT (GPLUS)
   LET (L= EX2)
   PRINT OUT (EX2=1.*L)
   LET (L= VAR)
   PRINT OUT (VAR=1.*L)
   LET (L= CV2)
```

```
      PRINT OUT (CV2=1.*L)
      PRINT OUT (CV=SQRT(CV2))
      LET (L=REPLACE(GPLUS,PI,3.141593))
      PRINT OUT (GPLUS=L)
      PRINT OUT(FPLUS=   GPLUS*EX/CV2)
    2 FORMAT('1  E(',I2,I2,' ) PROCESS MARGINAL INTERVAL DESNITY.' ///
     2      '     A(J) IS THE COEFFICIENT OF G(J,L)',/
     8      '     FI AND F ARE REPRESENTATIONS OF THE MARGINAL DENSITY',/
     3      '     SUMA IS THE SUM OF ALL A(J). ( SUMA SHOULD EQUAL 1)'/,
     4      '     EX IS THE EXPECTED INTERVAL LENGTH. ( EX =K/(P*R))',/
     5      '     EX2 IS THE EXPECTED SQUARED INTERVAL',/
     6      '     VAR IS THE INTERVAL VARIANCE '/,
     7      '     CV2 IS THE COEFFICIENT OF VARIATION SQUARED ',/
     8      '      GPLUS IS THE COUNT SPECTRUM INITIAL VALUE'/,
     9      '     FPLUS IS THE INTERVAL SPECTRUM INITIAL VALUE'//)
      IF (NPLOT.EQ.0)GO TO 75
      READ(5,3)N,XMAX
      WRITE (6,601)N, XMAX
    7 FORMAT (6A8)
    8 FORMAT (12A4)
      P=FLOAT(K)/FLOAT(NP)
      IF (NY.LE.1)
     2READ (5,7)(L3(J),J=1,12)
      DELTA=XMAX/FLOAT(N)
      NN=N+1
      LET (V='R')
      LET (T=REPLACE(F,R,V,E,2.718282))
      DO 90 L=1,NN
      Z=DELTA*FLOAT(L-1)
      X(L)=Z
      LET (U='Z')
      LET (F=REPLACE(T,X,U))
      Y(L)=ARITH(F)
   90 CONTINUE
      CALL LPLOT(Y,NN,0.,1.,NA,L1)
      WRITE(6,5)(Y(J),J=1,NN)
      NTEMP=NY
      IF(NT.EQ.0)GO TO 93
      NY=2
      IF(NTEMP .LE.1)NY=1
   93 CONTINUE
      CALL DRAW(NN,X,Y,NY,0,LABEL,L3    ,.5,.2,0,0,0,0,6,8,0,LAST)
      WRITE(6,9) LAST
      NY=3
      NT=NT-1
      IF (NTEMP.EQ.1.OR.NTEMP.EQ.2)NY=2
      IF (NT.GE.0)GO TO 93
    9 FORMAT ( '    LAST=',I2)
```

```
    3 FORMAT (I5,F10.0,F10.0)
    4 FORMAT(20A4)
    5 FORMAT(6E15.5)
      GO TO 75
   98 CONTINUE
      IF (NPLOT .EQ. 0) STOP
      READ(5,3)N,XMAX
      WRITE (6,601)N, XMAX
  601 FORMAT ('    N=',I5,'       XMAX=',E10.3)
      DELTA=XMAX/29.
      DO 95J=1,30
      T=DELTA*FLOAT(J-1)
      X(J)=T
      Y(J)=EXP(-T)
   95 CONTINUE
      MC=NY
   96 CONTINUE
      IF (NT.GT.0)MC=2
      CALL DRAW(30,X,Y,MC,3,LABEL,L3,1.,.2,0,0,0,0,6,8,0,LAST)
      WRITE (6,9) LAST
      NT=NT-1
      MC=NY
      IF(NT.GE.0)GO TO 96
      GO TO 75
  120 CONTINUE
      KK=K-1
      NY=2
      S=1
      READ (5,3)N,XMAX
      WRITE (6,601)N, XMAX
      NTT=NT
      DELTA=XMAX/29.
      DO 130 J=1,KK
  130 S=S*FLOAT(J)
      U=FLOAT(K)
      A=U**K/S
      DO 140 J=1,30
      T=DELTA*FLOAT(J-1)
      X(J)=T
      Y(J)=A*T**(KK)*EXP(-T*U)
  140 CONTINUE
      NY=2
  141 CONTINUE
      CALL DRAW(N,X,Y,NY,4,LABEL,L3    ,1.,2.,0,0,0,0,6,8,0,LAST)
      WRITE (6,9) LAST
      NTT=NTT-1
      IF (NTT.GE.0)GO TO 141
      DO 150 J=1,8
```

```
      X(J)=0.
      Y(J)=FLOAT(J-1)/FLOAT(J)
  150 CONTINUE
      N=9
      X(N)=0.
      Y(N)=1.
      NTT=NT
      NY=2
  151 CONTINUE
      IF (NTT.LE.0)NY=3
      CALL DRAW(N,X,Y,NY,2,LABEL,L3    ,1.,.2,0,0,0,0,6,8,0,LAST)
      WRITE (6,9) LAST
      NTT=NTT-1
      IF (NTT.GE.0)GO TO 151
   99 STOP
      END
      SUBROUTINE LPLOT
      RETURN
      END
```

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